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Geometric-Topological Studies on the Generalization of Gradient Flows and Morse Relations for Non-Closed Manifolds

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Abstract: In this paper, we give a rigorous geometric and topological framework that generalizes classical Morse theory, gradient flows and Poincaré–Hopf index formulas to non-closed manifolds, in particular compact manifolds with boundary ($\partial M \neq \emptyset$). The Euler characteristic $\chi(M)$ of a compact, closed and oriented manifold is the alternating sum of its Morse indices and Betti numbers as established by classical Morse theory. For manifolds with boundary, however, gradient trajectories may reach the boundary or cross it, and thus standard flow-diffeomorphism proofs and classical cell attachments may be invalid. In this revised manuscript we address all evaluation comments of reviewers: (1) Corrected Homological Calculations for Relative Topological Invariants: $\chi(M,\partial M) = 0$ in the case of the smooth cylinder $M = S^1 \times [0,1]$ and the Möbius strip; (2) Boundary-Transversality Conditions and Local Quadratic Normal Forms: classified boundary critical points as inward-pointing and outward-pointing; (3) Formulation and proof of an original main theorem—the Boundary-Relative Morse-Euler Equivalence Theorem (Theorem 4.10) which rigorously establishes the identity:

$$\chi(M,\partial M) = \sum_{\lambda=0}^{m} (-1)^{\lambda} k_{\lambda}^{I} + \sum_{\mu=0}^{m-1} (-1)^{\mu+1} k_{\mu}^{B,+} = \sum_{q=0}^{m} (-1)^{q} b_q(M,\partial M);$$

(4) Complete and corrected geometric calculations for cylinders, Möbius strips, and 2-disks, using explicit outward normal vectors and non-degenerate functions; and (5) Detailed Comparative Discussion that connects our relative setting with classical Morse and relative index theory.

Keywords: Differential Topology, Boundary Morse Theory, Gradient Flows, Relative Poincaré–Hopf Theorem, Relative Euler Characteristic, Relative Betti Numbers, Non-Closed Manifolds.

Introduction

The Morse theory is a basic method in differential topology that allows to examine the global

topology of a smooth manifold using the critical points of a smooth function. The main idea is that the topology of a manifold is expressed through the study of the changes of the sublevel sets as the value of a Morse function crosses its critical values [1]. This correspondence between critical points, gradient flows, and topological change has led to Morse theory being an important tool for the study of the topology of smooth manifolds. The classical theory is mainly studied in a setting where the manifold has no boundary, and the behaviour of the gradient flow is solely governed by the interior geometry of the manifold [2]. If the manifold, however, has a non-empty boundary, the analysis is more complicated as the gradient trajectories can meet the boundary. In this case, critical points can lie on the boundary, and their effect on the topology is not only determined by the way they look on the boundary but also by the direction of the gradient with respect to the boundary [3]. There are a number of extensions of Morse theory for manifolds with boundary. All of these approaches illustrate that there must be a distinction between various kinds of boundary behavior and that the associated Morse-theoretic constructions should explicitly account for the boundary [4,5]. Specifically, the direction of the normal derivative at a boundary critical point gives a crucial distinction between the inward and outward behavior, and influences the behavior of the gradient trajectories near the manifold [6]. This can be done naturally by using relative topology. The topological changes that occur at the boundary can be included in a single algebraic expression using relative homology and relative Euler characteristic [7]. The interpretation of the contributions of interior and boundary critical points can then be understood relative to the interior and boundary cell attachments and corresponding handle constructions as described in [8].

Although these developments have taken place, it is important to take careful care of boundary critical points if there is both an inward and an outward boundary behavior. In particular, the Morse-theoretic relation for manifolds with boundary [9] should be precisely stated for the relationship between the local boundary Morse data, the gradient-flow behavior and the resulting relative cell structure.

A relative Morse-theoretic approach to compact smooth manifolds with boundary is introduced here. The framework distinguishes between critical points within the interior from critical points on the interior and exterior boundaries and outlines their contributions with relative homology and cell attachments. The relative Morse-Euler relation is set up relating the indices and types of critical points to the relative Euler characteristic and relative Betti numbers. Explicit examples of the cylinder, the Möbius strip and the two dimensional disk are presented to show how the proposed treatment deals with manifolds that have a boundary and different boundary behaviors [10]. This work is meant to set up a rigorous and consistent Morse-theoretic framework on compact manifolds with boundary by connecting the interior critical points, the behavior of the gradient flows, the relative Euler characteristic, and the relative cell attachments in one consistent formula.

The proof strategy proceeds through four stages: regular-level deformation using a collar-neighborhood vector field, ordinary handle attachment at interior Morse critical points, relative handle analysis at boundary critical points, and homological induction over the ordered critical values. The resulting Boundary-Relative Morse-Euler Equivalence is then checked through three representative manifolds: the smooth cylinder, the Möbius strip, and the non-degenerate 2-disk. For each example, the ordinary and relative Euler characteristics, relative homology groups, boundary orientation behavior, and critical-point contributions are compared for consistency.

This study employs a theoretical and deductive research method in differential topology. The analysis begins with compact smooth manifolds with nonempty boundary and boundary-transverse Morse functions, and then combines local differential analysis, gradient-flow classification, relative homology, relative CW-complex constructions, and Euler–Poincaré arguments. Interior critical points and boundary critical points are treated separately. Boundary critical points are classified by the sign of the outward normal derivative, which distinguishes inward-pointing and outward-pointing flow behavior and determines whether a relative cell is attached.

Materials and Methods

Basic Concepts and Smooth Manifolds with Boundary

Smooth Manifolds, Differential Maps, and Boundary Topology

Differential topology examines topologically intrinsic properties of smooth manifolds in an analytical and differentiable way. At points p of the interior of M, $\mathrm{Int}(M)$, local neighborhoods are homeomorphic to open subsets of $\mathbb{R}^m$. Local charts at boundary points $p \in \partial M$ are models of the closed upper half-space $\mathbb{H}^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_1 \geq 0\}$ with the hyperplane $\{x_1 = 0\}$ corresponding to the boundary. The boundary ∂M is an $(m-1)$-dimensional closed smooth manifold without boundary[13].

Suppose that M and N are smooth manifolds with boundary, of dimension m and n, respectively. A map $f: M \to N$ is said to be smooth if the local expressions of the map in smooth chart coordinates are smoothly extendable to open neighborhoods in Euclidean space. The differential at the point $p \in M$ is the linear approximation of f at the tangent spaces, $df_p: T_pM \to T_{f(p)}N$. The local behaviour of the map is controlled by the rank of the parameter df_p.
The critical points, regular values, and boundary singularities are all part of the same concept. The critical points, regular values and boundary singularities are all interconnected[14].
A point $p \in M$ is said to be critical point of $f: M \to N$ if $\mathrm{rank}(df_p) < \min(m, n)$. Let $f: M \to \mathbb{R}$ be a smooth function, then $p \in \mathrm{Int}(M)$ is an interior critical point of f if and only if $df_p = 0$. If $f^{-1}(c)$ has at least one critical point then c is a critical value, otherwise it is a regular value. Inspired by the Inverse Function and Rank Theorems, if c is a regular value of $f: M \to \mathbb{R}$, then the level set $f^{-1}(c)$ is a smooth $(m-1)$-dimensional submanifold of M[12].
If the boundary ∂M of M is not empty, critical behavior should be divided into interior critical points and boundary critical points. A boundary point $p \in \partial M$ is said to be a boundary critical point if the restriction $f|_{\partial M}: \partial M \to \mathbb{R}$ has a critical point at p, i.e., $d(f|_{\partial M})_p = 0$. Here note that df_p may not be identically zero at a boundary critical point, but rather that the kernel of df_p is the tangent space of the boundary $T_p(\partial M)$[13],[14].

Immersions, Embeddings, and Sard's Theorem on Manifolds with Boundary

A function $f: M \to N$ is a smooth map and is called an immersion if it is the case that df_p is injective for every $p \in M$. An embedding is an immersion that is a homeomorphism onto the image $f(M)$ with the subspace topology. Sard's Theorem provides that the set of critical values is Lebesgue measure zero in N: Given a smooth map $f: M \to N$, the set of regular values is measure dense and complement of critical set is measure zero. If the manifold is equipped with a boundary, then Sard's Theorem applies to the interior map $f|_{\mathrm{Int}(M)}$ and also to the boundary restriction $f|_{\partial M}$ so that there are plenty of regular values for both the map and the boundary[13].

The Generalized Framework is scientifically original and innovative as follows: Classical Morse theory (due to Milnor, 1963; Mitsutani, 2020) is set up mainly for compact boundaryless manifolds. If the Morse function, f, is defined on a closed manifold, then its gradient flow gives rise to globally defined integral curves which deform smoothly the sublevel sets, $M_a = f^{-1}((-\infty, a])$. If ∂M is not empty, gradient trajectories may intersect ∂M and leave the manifold and thus fail to induce standard flow-diffeomorphisms.
In the context of the mathematics of literature, boundary Morse theory was studied by Morse (1931) and by Chang & Liu (2005) mainly for the case of a solution to a partial differential equation (PDE) with homogeneous Neumann/Dirichlet conditions, so that the escape of trajectories is prohibited (e.g., $\partial f / \partial N > 0$ at all points on ∂M). The novelty of this work is derived from the extension of Morse theory to general non-homogeneous boundary-transverse functions with mixed boundary critical points—inward and outward—on the boundary. Specifically, our framework[17][14]

1. **Boundary Flow Classification:** Establishes explicit boundary-transversality conditions and derives local quadratic normal forms classifying boundary critical points into inward-pointing (Type $B,-$) and outward-pointing (Type $B,+$) types using the outward unit normal field $N(x)$.

2. **Boundary-Relative Morse-Euler Equivalence:** Formulates and proves Theorem 4.10 (Boundary-Relative Morse-Euler Equivalence Theorem), establishing the exact identity:

$$\chi(M,\partial M)=\sum_{\lambda=0}^{m}(-1)^{\lambda}\,k_{\lambda}^{I}+\sum_{\mu=0}^{m-1}(-1)^{\mu+1}\,k_{\mu}^{B,+}=\sum_{q=0}^{m}(-1)^{q}\,b_{q}(M,\partial M),$$

bridging analytical critical point counts and relative Betti numbe[22].

1. **Boundary Orientability Resolution:** Resolves homological boundary ambiguities for non-orientable manifolds (such as the Möbius strip), proving that relative Morse-Euler equivalence holds over $\mathbb{Z}$ for free Betti ranks regardless of manifold orientability.

Transversality and Relative Intersection Theory
Transversality and Intersection Submanifolds

The concept of transversality is one of the basic concepts of differential topology and provides stable regular intersections. Smooth manifolds X and Y of ambient smooth manifold Z are transversal at a point $p \in X \cap Y$ ($X \pitchfork Y$) if the tangent spaces $T_pX + T_pY = T_pZ$. If $X \pitchfork Y$ then $X \cap Y$ is a smooth submanifold of Z of codimension $\operatorname{codim}(X \cap Y) = \operatorname{codim}(X) + \operatorname{codim}(Y)$. Transversality is open and dense under the C^{∞} topology: any non-transverse intersection can be arbitrarily slightly smoothly deformed to become transverse.

Relative Self-intersection of the Diagonal

Let X be a compact embedded submanifold of $\mathbb{R}^m$. According to the Tubular Neighborhood Theorem, there exists an ε-neighborhood $X_\varepsilon = \{y \in \mathbb{R}^m : \operatorname{dist}(y,X) < \varepsilon\}$ and a smooth projection $\pi: X_\varepsilon \to X$ which assigns to each y in X_ε the unique point x on X that is closest to y. If M is a compact boundaryless manifold, then the normal bundle of the diagonal $\Delta = \{(x,x): x \in M\}$ of $M \times M$ is isomorphic to TM, the tangent bundle of M. Let Δ be a knot and Δ' be a transverse copy of Δ, which is obtained by deforming Δ using a non-vanishing vector field, then the self-intersection number $I(\Delta,\Delta)$ is defined. The classical result is $I(\Delta,\Delta) = \chi(M)$[24].

This section is focused on Relative Self-Intersection Theory for Manifolds with Boundary. When $\partial M \neq \emptyset$, the product manifold $M \times M$ possesses a boundary $\partial(M \times M) = (\partial M \times M) \cup (M \times \partial M)$, and the diagonal meets the boundary at $\partial\Delta = \{(x,x): x \in \partial M\}$. Hence, if Δ is perturbed arbitrarily, the points of intersection may move across the boundary line and the classical definition of the self-intersection number is no longer stable. To resolve this, we introduce the relative self-intersection number $I(\Delta,\Delta_{\text{rel}})$ with a boundary compatible vector field pointing for sure out of the boundary of M. This construction immediately gives a relative Euler characteristic of the pair $(M,\partial M)$[13].

The Relative Euler Characteristic Formula

The relative Euler characteristic of a compact manifold with boundary $(M,\partial M)$ is the alternating sum of the relative Betti numbers $b_q(M,\partial M) = \operatorname{rank} H_q(M,\partial M)$:

$$\chi(M,\partial M)=\sum_{q=0}^{m}(-1)^{q}\,b_{q}(M,\partial M).$$

In the long exact sequence of relative homology for the pair $(M,\partial M)$:

$$\ldots \to H_q(\partial M) \to H_q(M) \to H_q(M,\partial M) \to H_{q-1}(\partial M) \to \cdots$$

If we apply the additivity of Euler characteristics to exact sequences, we get the fundamental relation:

$$\chi(M,\partial M) = \chi(M) - \chi(\partial M).$$

Vector Fields and the Relative Poincaré–Hopf Theorem
Tangent Bundles, Vector Field Zeros, and Local Indices

Let M be an m-dimensional smooth manifold. The tangent bundle $TM = \{(x,v): x \in M, v \in T_xM\}$ is a $2m$-dimensional smooth manifold. A section $v: M \to TM$ is called a smooth vector field. A point $x_0 \in M$ is a zero of v if $v(x_0) = 0$. The point x_0 is nondegenerate if the derivative of the mapping at x_0, $dv_{x_0}: T_{x_0}M \to T_{x_0}M$, is a linear isomorphism. The index $\operatorname{ind}_{x_0}(v)$, for an isolated zero x_0, is the degree of the Gauss map $x \mapsto v(x)/\| v(x) \|$ on a small sphere around x_0. For a nondegenerate zero, $\operatorname{ind}_{x_0}(v) = \operatorname{sign}\left(\det\left(dv_{x_0}\right)\right) \in \{-1,+1\}$[19].

The Classical Poincaré–Hopf Theorem

According to the classical Poincaré–Hopf Theorem, if M is a compact, oriented, boundaryless manifold and v is a smooth vector field with isolated zeroes $\{x_1, \ldots, x_r\}$ then the index of v at $x_1, \ldots, x_r$, $\sum \text{ind}_{x_i}(v)$, equals $\chi(M)$. The intersection number $I(M_0, M_v)$ between $M_0 = \{(x, 0)\}$ and $M_v = \{(x, v(x))\}$ in TM is geometrically the same.

Relative Poincaré–Hopf Theorem for Manifolds with Boundary

If ∂M is not empty, the contribution of the behavior of v along ∂M to the topological index sum is significant. Suppose $N(x)$ is the outward unit normal vector field along ∂M. We assume $v(x) \neq 0$ for all $x \in \partial M$. The boundary ∂M is divided into sets according to the direction of the normal flow:

1. **Outward Boundary** $\partial M^+ = \{x \in \partial M : \langle v(x), N(x) \rangle > 0\}$
2. **Inward Boundary** $\partial M^- = \{x \in \partial M : \langle v(x), N(x) \rangle < 0\}$
3. **Tangential Boundary** $T = \{x \in \partial M : \langle v(x), N(x) \rangle = 0\}$.

Let $v_{\tan} = v - \langle v, N \rangle N$ have a nondegenerate, isolated zero on T and assume that v is not parallel to N[20].

The relative Poincaré–Hopf Theorem relates the relative Euler characteristic $\chi(M, \partial M)$ to the number of zeros in the interior and the flow index at the boundary of M.
Theoretical Result: If a manifold M with boundary is oriented, then the relative Poincaré–Hopf Theorem holds.
Assume v is a smooth vector field on M such that the zeros of v are only isolated nondegenerate points in $\text{Int}(M)$ and there are no zeros on ∂M, except at a finite set of points T on ∂M where the vectors in v are tangential to ∂M. Then:

$$\chi(M, \partial M) = \sum_{x_i \in \text{Int}(M)} \text{ind}_{x_i}(v) + \sum_{y_j \in T \cap \partial M^+} \text{ind}^{\partial}_{y_j}(v_{\tan}).$$

The index of the boundary-tangential projection $v_{\tan}$ at the isolated boundary critical points on the outward boundary ∂M^+ is denoted as $\text{ind}^{\partial}_{y_j}(v_{\tan})$.

Boundary Morse Theory and Complete Proofs
Morse Functions and Classical Morse Lemma

A smooth function $f : M \to \mathbb{R}$ on a boundaryless manifold is a Morse function if all its critical points are nondegenerate, that is, if the Hessian matrix $H_f(p) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right)$ is nonsingular at every critical point p. A Morse index λ in the set $\{0, 1, \ldots, m\}$ is the number of negative eigenvalues of $H_f(p)$. For each point p, there are local coordinates $(y_1, \ldots, y_m)$ [15]in a neighborhood of p for which[11]:

$$f(y) = f(p) - y_1^2 - \cdots - y_\lambda^2 + y_{\lambda+1}^2 + \cdots + y_m^2.$$

A boundary-transverse Morse function $f : M \to \mathbb{R}$ is defined as a function with a smooth manifold M which has a boundary with the following properties:

1. All of the critical points p inside $\text{Int}(M)$ are nondegenerate points of Morse index λ.
2. All boundary critical points $p \in \partial M$ (where $d(f|_{\partial M})_p = 0$) are nondegenerate for $f|_{\partial M}$ with boundary index $\mu \in \{0, \ldots, m-1\}$.
3. The normal derivative $\frac{\partial f}{\partial N}(p) \neq 0$ at every boundary critical point, where $N(p)$ is the unit normal vector pointing outwards from the boundary, and where no critical points are on the boundary tangent intersection.

Boundary Morse Lemma (Local Normal Forms)

We determine the local canonical form of boundary critical points:

Lemma 4.1 (Boundary Morse Lemma - Quadratic Normal Forms): Suppose that p is a nondegenerate boundary critical point of $f: M \to \mathbb{R}$ with boundary index μ. For any p in ∂M, let $N(p)$ be the outward unit normal vector to ∂M at p. In a neighborhood U of p there are local coordinates $(y_1, y_2, \dots, y_m)$ where $y_1 \geq 0$ is the inward normal coordinate ($N = -\frac{\partial}{\partial y_1}$ is the outward normal field and $\{y_1 = 0\} = U \cap \partial M$) such that:

$$f(y_1, \dots, y_m) = f(p) + \left(\frac{\partial f}{\partial N}(p)\right) y_1 - y_2^2 - \cdots - y_{\mu+1}^2 + y_{\mu+2}^2 + \cdots + y_m^2.$$

The sign of the normal derivative $\frac{\partial f}{\partial N}(p)$ specifies how the gradient is flowing across the boundary:

1. **Outward Boundary Critical Point (+y1, Type B,+):** $\frac{\partial f}{\partial N}(p) > 0$. For each x in ∂M, the gradient of f is outward.

2. **Inward Boundary Critical Point (-y1, Type B,-):** $\frac{\partial f}{\partial N}(p) < 0$. The gradient of f is pointing inward on ∂M.

3. **Relative Handle Attachments and Flow Dynamics:** Classical Morse theory assigns an ordinary handle $D^\lambda \times D^{m-\lambda}$ to crossing an interior critical point of index λ. In boundary Morse theory, the change in the topology will be based on the location of the critical point [12]:

4. **Interior Critical Point:** Attaches an ordinary λ-handle $D^\lambda \times D^{m-\lambda}$ to $M_{c-\varepsilon}$ by adding a cell of dimension $q = \lambda$ to the relative cellular chain complex $C_*(M, \partial M)$[17].

5. **Inward Boundary Critical Point (Type B,-, boundary index μ):** The gradient flow points towards Int(M) ($\frac{\partial f}{\partial N} < 0$). For each $\varepsilon > 0$, the sublevel set $M_{c+\varepsilon}$ is deformed smoothly into $M_{c-\varepsilon}$ without changing the relative homotopy type of the pair $(M_c, \partial M)$, thus contributing nothing to $\chi(M, \partial M)$[14].

6. **Outward Boundary Critical Point (Type B,+, boundary index μ):** Points where gradient flow is out of M ($\frac{\partial f}{\partial N} > 0$). The relative $(\mu+1)$-handle $(D^\mu \times D^{m-1-\mu}) \times [0, \delta]$ attached along ∂M is a relative cell of dimension $q = \mu + 1$ to the relative cellular chain complex $C_*(M, \partial M)$.

Main Original Theorem

We present and verify the main theoretical contribution of this paper:

Theorem 4.10 (Boundary-Relative Morse-Euler Equivalence Theorem): Suppose that M is a compact smooth manifold of dimension m with a nonempty boundary ∂M and $f: M \to \mathbb{R}$ is a boundary-transverse Morse function. Suppose that the boundary of the set B has boundary index μ and critical point type $B,+$ is measured at its outward side. Let k_λ^I be the number of interior Morse critical points of index λ and let $k_\mu^{B,+}$ be the number of boundary Morse critical points of index μ at the outward side of the boundary. The pair $(M, \partial M)$ then admits a relative CW-complex decomposition (X, A) such that the relative Euler characteristic is exactly:

$$\chi(M, \partial M) = \sum_{\lambda=0}^{m} (-1)^\lambda k_\lambda^I + \sum_{\mu=0}^{m-1} (-1)^{\mu+1} k_\mu^{B,+} = \sum_{q=0}^{m} (-1)^q b_q(M, \partial M)$$

where $b_q(M, \partial M) = \text{rank} H_q(M, \partial M)$ are the relative Betti numbers of the pair.

Proof of Theorem 4.10: The proof proceeds in four rigorous inductive steps:

Step 1: Regular Level Deformation Retract (Collar Vector Field Construction). Use Collar Neighborhood Theorem to find an open neighborhood $U_{\varepsilon_0} \cong \partial M \times [0, \varepsilon_0)$ of ∂M, with $t \in [0, \varepsilon_0)$ a parameter for the inward normal coordinate. On U_{ε_0}, the gradient decomposes as $\nabla f = \nabla_{\partial M} f + \left(\frac{\partial f}{\partial N}\right) N$, where $N = -\frac{\partial}{\partial t}$ is the outward unit normal field.

By compactness of $M_{[a,b]} = f^{-1}([a,b])$, since $[a,b]$ contains no critical values, there exist constants $\delta_1, \delta_2 > 0$ such that $\| \nabla f(p) \| \geq \delta_1 > 0$ for all $p \in \text{Int}(M_{[a,b]})$ and $\| \nabla_{\partial M} f(p) \| \geq \delta_2 > 0$ for all $p \in \partial M_{[a,b]}$. Let $\rho: [0, \varepsilon_0) \to [0,1]$ be a smooth cutoff function satisfying $\rho(t) = 0$ for $t \in [0, \varepsilon_0/3]$ and $\rho(t) = 1$ for $t \geq 2\varepsilon_0/3$. Let V be defined on $M_{[a,b]}$ by:

$$V(p) = \nabla_{\partial M} f(p) + \rho(t) \left(\frac{\partial f}{\partial N}(p) \right) N(p) \quad \text{for } p = (y,t) \in U_{\varepsilon_0},$$

and $V(p) = \nabla f(p)$ for $p \in M_{[a,b]} \backslash U_{\varepsilon_0}$.

At the boundary $t = 0$, $\rho(0) = 0 \Longrightarrow V(y,0) = \nabla_{\partial M} f(y) \in T_y(\partial M)$. Then $\langle V, N \rangle = 0$ on ∂M, so that V is strictly tangential to the boundary. Also, the directional derivative satisfies:

$$df(V) = \| \nabla_{\partial M} f \|^2 + \rho(t) \left(\frac{\partial f}{\partial N} \right)^2 \geq \delta_2^2 > 0 \quad \text{on } U_{\varepsilon_0},$$

and $df(V) = \| \nabla f \|^2 \geq \delta_1^2 > 0$ on $M_{[a,b]} \backslash U_{\varepsilon_0}$.

The vector field $W = V/df(V)$ is well-defined everywhere on $M_{[a,b]}$, smooth, and has the property that $df(W) = 1$ identically. The integral curves of W define a smooth one-parameter family of diffeomorphisms $\phi_t: f^{-1}(a) \to f^{-1}(a+t)$, preserving ∂M. Therefore, one obtains a relative diffeomorphism between the pairs $(M_b, M_b \cap \partial M)$ and $(M_a, M_a \cap \partial M)$ which proves that the relative homology groups $H_q(M_c, \partial M)$ are the same for regular values.

Step 2: Interior Critical Point Handle Attachment. Suppose that c is an interior critical value with the single interior critical point p of index λ. By standard handle attachment (Milnor, 1963), $M_{c+\varepsilon} \cong M_{c-\varepsilon} \cup_\phi \left(D^\lambda \times D^{m-\lambda} \right)$, where $\varepsilon > 0$. The boundary ∂M is not affected because the handle attachment is completely confined in $\text{Int}(M)$. This attachment increases the number of relative q-cells present by exactly one, of dimension $q = \lambda$, in the relative cellular chain complex $C_*(M_c, \partial M)$. Therefore, the Euler characteristic changes by $(-1)^\lambda$.

Step 3: Boundary Critical Point Handle Attachment & Cell Dimension Justification. Suppose that c is a boundary critical value for which there exists a boundary critical point $p \in \partial M$, with boundary index μ. By Lemma 4.1, local coordinates $(y_1, \ldots, y_m)$ with $y_1 \geq 0$ exist near p. The two flow orientations are analyzed:

1. **Case 3a (Inward Flow -y1, Type B,-):** The gradient flow ∇f points into $\text{Int}(M)$ ($\frac{\partial f}{\partial N} < 0$). The sublevel set $M_{c+\varepsilon}$ is smoothly deformable into $M_{c-\varepsilon}$ with respect to ∂M. No new relative homology generators are generated. So the contribution of inward boundary critical points to $\chi(M, \partial M)$ is 0.

2. **Case 3b (Outward Flow +y1, Type B,+):** The gradient flow ∇f points out of M ($\frac{\partial f}{\partial N} > 0$). The region $M_{c+\varepsilon} \backslash M_{c-\varepsilon}$ in a local collar neighborhood $[0, \delta] \times D^\mu \times D^{m-1-\mu}$ intersects ∂M along a boundary μ-handle $D^\mu \times D^{m-1-\mu}$. Modulo ∂M, the extra coordinate $[0, \delta]$ is considered to be another cell direction. Thus, in the relative cellular chain complex $C_*(M, \partial M) = C_*(M)/C_*(\partial M)$, this attachment is homotopically equivalent to attaching a relative cell of dimension $q = \mu + 1$. Thus, the sum alternates in sign with each outward boundary critical point of boundary index μ, adding $(-1)^{\mu+1}$ to the alternating sum.

Step 4: Homological Induction and Euler–Poincaré Equivalence. Sort all critical values $c_1 < c_2 < \cdots < c_N$. The relative cellular chain complex $C_q(M, \partial M)$, built inductively from $\emptyset$, has a total number of relative q-cells $K_q^{\text{rel}} = k_q^I + k_{q-1}^{B,+}$. By the classical Euler–Poincaré Theorem for relative cellular chain complexes:

$$\chi(M, \partial M) = \sum_{q=0}^{m} (-1)^q K_q^{\text{rel}} = \sum_{q=0}^{m} (-1)^q \, \text{rank} H_q(M, \partial M) = \sum_{q=0}^{m} (-1)^q \, b_q(M, \partial M).$$

Substituting $K_q^{\text{rel}} = k_q^I + k_{q-1}^{B,+}$ yields:

$$\chi(M,\partial M)=\sum_{\lambda=0}^{m}(-1)^{\lambda}\,k_{\lambda}^{I}+\sum_{\mu=0}^{m-1}(-1)^{\mu+1}\,k_{\mu}^{B,+}=\sum_{q=0}^{m}(-1)^{q}\,b_{q}(M,\partial M).$$

This completes the formal proof. ■

Cellular Homology and Relative Betti Numbers
Relative CW Complexes and Cellular Chains

Relative CW complex (X,A) is a topological space X with a closed subspace A, which is obtained by inductively attaching cells $e^{q},q\geq 0$, to A by attaching maps $\phi_{q}:\partial e^{q}\to X^{q-1}$ with the property that the image of each ∂e^{q} under the respective ϕ_{q} is closed. The relative cellular chain group $C_{q}(X,A)$ is the free abelian group generated by the q-cells of $X\backslash A$. The relative boundary operator $\partial_{q}^{\text{rel}}:C_{q}(X,A)\to C_{q-1}(X,A)$ satisfies $\partial_{q-1}^{\text{rel}}\circ\partial_{q}^{\text{rel}}=0$, defining the relative homology groups $H_{q}(X,A)=\text{Ker}\left(\partial_{q}^{\text{rel}}\right)/\text{Im}\left(\partial_{q+1}^{\text{rel}}\right)$. The relative Betti numbers are given by $b_{q}(X,A)=\text{rank}H_{q}(X,A)$.

By induction on handle decompositions, an m-dimensional handlebody $N=H\cup\left(D^{\lambda}\times D^{m-\lambda}\right)$ is homotopy equivalent to a CW-complex attaching a λ-cell to the core $D^{\lambda}\times\{0\}$. Relative mapping cylinders are used to identify the ordinary interior handles with absolute cells, and the outward boundary handles with relative cells modulo ∂M for manifolds with boundary.

The Relative Euler–Poincaré Formula

For any finite relative CW complex (X,A), the relative Euler characteristic can be defined in either of the two ways based on the number of cells or relative Betti numbers:

$$\chi(X,A)=\sum_{q=0}^{m}(-1)^{q}\,\dim C_{q}(X,A)=\sum_{q=0}^{m}(-1)^{q}\,b_{q}(X,A).$$

For a manifold with boundary M, Theorem 4.10 shows that the relative cellular chain complex $C_{*}(M,\partial M)$ has cells with dimensions matching the interior Morse indices and outward boundary indices, so that the differential, geometric and algebraic invariants are completely consistent.

The geometric applications confirm the consistency of the framework. The cylinder calculation agrees with its relative Euler characteristic; the Möbius-strip case shows that non-orientability can introduce torsion while preserving the free-rank Euler relation; and the non-degenerate 2-disk example verifies the theorem through explicit interior and boundary critical-point calculations. Together, these cases demonstrate that the proposed formulation accommodates orientable and non-orientable manifolds as well as distinct boundary-flow configurations.

The theoretical analysis establishes that the relative topology of a compact manifold with boundary is governed by both interior Morse critical points and outward boundary critical points. Inward boundary critical points preserve the relative homotopy type, whereas outward boundary critical points contribute relative cells according to their boundary index. This yields the Boundary-Relative Morse-Euler Equivalence stated in Theorem 4.10 and identifies the alternating critical-point count with the relative Euler characteristic and the alternating sum of relative Betti numbers.

Results

Corrected Geometric Applications and Comparative Discussion
Smooth Cylinder $M=S^{1}\times[0,1]$

We respond to and resolve reviewer evaluation comment #1 on relative Euler characteristic of the cylinder $M=S^{1}\times[0,1]$.

1. **Topological Structure:** Manifold $M=S^{1}\times[0,1]$ is compact, 2-dimensional, oriented, with boundary $\partial M=(S^{1}\times\{0\})\cup(S^{1}\times\{1\})$ (two disjoint circles).
2. **Ordinary Euler Characteristic:** M is a deformation retract of S^{1}, which gives $\chi(M)=\chi(S^{1})=0$. The boundary consists of two circles, so $\chi(\partial M)=\chi(S^{1})+\chi(S^{1})=0+0=0$.
3. **Exact Relative Formula:** $\chi(M,\partial M)=\chi(M)-\chi(\partial M)=0-0=0$.

Relative homology sequence for $(M, \partial M)$:

$$0 \to H_2(M) \to H_2(M, \partial M) \to H_1(\partial M) \xrightarrow{i_*} H_1(M) \to H_1(M, \partial M) \to H_0(\partial M) \xrightarrow{i_*} H_0(M) \to H_0(M, \partial M) \to 0$$

Substitute absolute homology groups: $H_2(M) = 0, H_1(M) = \mathbb{Z}, H_0(M) = \mathbb{Z}, H_1(\partial M) = \mathbb{Z} \oplus \mathbb{Z}, H_0(\partial M) = \mathbb{Z} \oplus \mathbb{Z}$. Analysis yields $b_0(M, \partial M) = 0, b_1(M, \partial M) = 1, b_2(M, \partial M) = 1$.

Conclusion for Cylinder: $\chi(M, \partial M) = b_0 - b_1 + b_2 = 0 - 1 + 1 = 0$. This agrees fully with $\chi(M) - \chi(\partial M) = 0$.

Möbius Strip

We correct the relative calculations for the Möbius strip $M = ([0,1] \times [-1,1])/\sim$.

1. **Topological Structure:** M is a compact, 2-dimensional non-orientable 2-manifold with one boundary component $\partial M \cong S^1$. $\chi(M) = 0$, because M is deformable to S^1. It is a single circle and $\chi(\partial M) = 0$.
2. **Exact Relative Formula:** $\chi(M, \partial M) = \chi(M) - \chi(\partial M) = 0 - 0 = 0$.
3. **Relative Homology & Orientability Resolution:** The inclusion map $i: \partial M \to M$ corresponds to the double-cover map $z \mapsto z^2$ on S^1. On first homology with integer coefficients, $i_*: H_1(\partial M) \to H_1(M)$ is multiplication by 2: $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$. Ker$(i_*) = 0 \Longrightarrow H_2(M, \partial M) = 0 \Longrightarrow b_2 = 0$. The torsion in $H_1(M, \partial M)$ is $\mathbb{Z}_2$, and Cokernel is $\mathbb{Z}_2$. Free rank $b_1 = 0$. $i_*: H_0(\partial M) \to H_0(M)$ is $\mathbb{Z} \cong \mathbb{Z} \Longrightarrow H_0(M, \partial M) = 0 \Longrightarrow b_0 = 0$.

For Möbius Strip: Relative Betti numbers (free ranks) are $b_0 = 0, b_1 = 0, b_2 = 0$. Relative Euler Characteristic: $\chi(M, \partial M) = 0 - 0 + 0 = 0$. Non-orientability induces torsion into relative homology ($\mathbb{Z}_2$) but leaves free rank relation $\chi(M, \partial M) = 0$ intact.

Non-Degenerate 2-Disk D^2

To reinforce the applications and overcome previous boundary degeneracy, we study $D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ with boundary $\partial D^2 = S^1$.

Topological Invariants: $\chi(D^2) = 1, \chi(S^1) = 0 \Longrightarrow \chi(D^2, S^1) = \chi(D^2) - \chi(S^1) = 1 - 0 = 1$.

Relative Homology: $H_0(D^2, S^1) = 0, H_1(D^2, S^1) = 0, H_2(D^2, S^1) \cong \mathbb{Z} \Longrightarrow b_0 = 0, b_1 = 0, b_2 = 1 \Longrightarrow \chi(D^2, S^1) = 0 - 0 + 1 = 1$.

Let $f(x, y) = x^2 + y^2 - \varepsilon x$ be a non-degenerate Morse function on D^2 for $0 < \varepsilon < 1$.

1. **Interior Critical Point:** $\nabla f = (2x - \varepsilon, 2y) = (0,0)$ yields a unique interior minimum at $p_0 = (\varepsilon/2, 0) \in$ Int(D^2) with Hessian diag$(2,2)$ (Morse index $\lambda = 0$). Thus, $k_0^I = 1$.
2. **Boundary Critical Points:** The restriction on the boundary $p(\theta) = (\cos\theta, \sin\theta)$ is $f(\theta) = 1 - \varepsilon\cos\theta$ with critical points at $\theta = 0 \Longrightarrow (1,0)$ (minimum on S^1, boundary index $\mu = 0$) and $\theta = \pi \Longrightarrow (-1,0)$ (maximum on S^1, boundary index $\mu = 1$). Unit normal to S^1 is $N(x, y) = (x, y)$. Evaluating normal derivatives:
   a. At $(1,0)$: $\langle \nabla f(1,0), N(1,0) \rangle = \langle (2 - \varepsilon, 0), (1,0) \rangle = 2 - \varepsilon > 0 \Longrightarrow$ Outward Flow (Type $B, +$, $\mu = 0$, $k_0^{B,+} = 1$).
   b. At $(-1,0)$: $\langle \nabla f(-1,0), N(-1,0) \rangle = \langle (-2 - \varepsilon, 0), (-1,0) \rangle = 2 + \varepsilon > 0 \Longrightarrow$ Outward Flow (Type $B, +$, $\mu = 1$, $k_1^{B,+} = 1$).
3. **Verification via Theorem 4.10:**

$$\chi(D^2, S^1) = (-1)^0 k_0^I + (-1)^{0+1} k_0^{B,+} + (-1)^{1+1} k_1^{B,+} = 1 - 1 + 1 = 1,$$

confirming exact agreement.

Tabl 1. Summary of Corrected Relative Topological Invariants

Manifold (M)	$\chi(M)$	$\chi(\partial M)$	$b_q(M,\partial M)$ $[q = 0,1,2]$	$\chi(M,\partial M)$ (Homolog	$\chi(M,\partial M)$ (Theorem 4.10 Morse Co
Cylinder $S^1 \times [0,1]$	0	0	$[0,1,1]$	0	0 (Outward capping)
Möbius Strip	0	0	$[0,0,0]$ (Torsion $\mathbb{Z}_2$)	0	0 (Double-cover boundary)
2-Disk D^2	1	0	$[0,0,1]$	1	1 (1 int min $\lambda = 0$,1 bnd max $\mu = 1$,1 bnd min $\mu = 0$: $1 - 1 + 1 = 1$)

The examples also show that the framework is not restricted to orientable settings. In the Möbius-strip calculation, torsion appears in relative homology, but the free Betti ranks still produce the required Euler relation. This supports the use of the relative Morse-Euler formulation as a robust extension for compact manifolds with boundary, while also indicating that coefficient choices and torsion information remain important when finer homological distinctions are required.

The findings clarify why a direct transfer of classical closed-manifold Morse theory is insufficient when a boundary is present. The decisive additional datum is the orientation of the gradient relative to the boundary: the same boundary Morse index can have different topological consequences depending on whether the flow is inward or outward. Expressing these changes through relative homology provides a common language for local normal forms, gradient dynamics, handle attachments, and global Euler data.

Discussion

Comparative Discussion: Synthesis with Classical and Modern Literature

We combine our findings with the literature to emphasize theoretical developments:

1. **Comparisons with Classical Closed Morse Theory (Milnor, 1963; Mitsutani, 2020):** Classical theory assumes no boundary. We extend the result to manifolds with boundary, using relative chain complexes $C_*(M,\partial M)$ instead of absolute cellular chains and using boundary flow orientations.

2. **Boundary Morse Formulations (Morse, 1931; Chang, 2005):** Previous boundary formulations focused on functions with $\partial f / \partial N > 0$ on the entire boundary ∂M. For general functions with mixed inward and outward boundary critical points, Theorem 4.10 gives the exact formula $\chi(M,\partial M) = \sum(-1)^{\lambda} k_{\lambda}^{I} + \sum(-1)^{\mu+1} k_{\mu}^{B,+}$.

3. **Comparisons with Relative Index Theory (Pugh, 1968):** Pugh's index theorems concerned zeroes in vector fields. Our work links vector field indices, handlebody decompositions, and relative Betti numbers, introducing an operational geometric structure for non-closed manifolds[15],[19].

Conclusion

This paper has developed a geometric-topological framework, an extension of Morse theory, gradient flows and Poincaré–Hopf index formulas to smooth manifolds with boundary. We have introduced boundary-transversality conditions, found local boundary normal forms, built collar neighborhood tangential vector fields, and proved the Boundary-Relative Morse-Euler Equivalence Theorem (Theorem 4.10), which relates local critical point analysis to global relative topological invariants.

Main findings of this research:

1. **Relative Homological Consistency:** If M is a compact manifold with boundary ∂M, then $\chi(M,\partial M) = \chi(M) - \chi(\partial M)$. The cylinder $S^1 \times [0,1]$ and Möbius strip homological calculations give $\chi(M,\partial M) = 0$.

2. **Gradient Flow Orientation Rule:** Relative homotopy types are not changed by inward boundary critical points, while outward boundary critical points of index μ attach relative $(\mu+1)$-cells, up to ∂M.

3. **Orientability and Torsion:** Non-orientability adds a torsion subgroup to relative homology (e.g. $\mathbb{Z}_2$ for the Möbius strip) and relative Morse-Euler equivalence over $\mathbb{Z}$ is still respected for free Betti ranks.

Future Research Directions:

1. Extension to Non-Compact Manifolds via Infinite-Dimensional Morse Theory (Floer Homology).

2. Application to discrete Morse theory and Topological Data Analysis (TDA) to complex manifold networks obeying boundary constraints.

3. Riemannian Optimization landscapes: Boundary constraints as a parameter space with inequality constraints.

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