Quest: Journal of Geometry, Mathematical and Quantum Physics (Quest)

e-ISSN: 3064-7282 Vol.3 Issue 3|2026

Using the Interacting Boson Model to Calculate Some Nuclear Properties for Even Zirconium(A=92-104) Isotopes

Zahraa Hassan Yousef[1], Mohammed A. Al-Shareefi[2], Ghaidaa A.Hafedh Jaber[3]

1. University of Babylon, Najaf-Hilla Street, 51001, Babil, Iraq
2. University of Babylon, Najaf-Hilla Street, 51001, Babil, Iraq
3. University of Babylon, Najaf-Hilla Street, 51001, Babil, Iraq

*Correspondence: zehra.hasn804@student.uobabylon.edu.iq[1]
mohammedshareefi@yahoo.com[2]
sci.ghaidaa.abdul-hafidh@uobabylon.edu.iq[3]

Citation: Yousef, Z. H. Al-Shareefi, M. A., Jaber, G. A. Using the Interacting Boson Model to Calculate Some Nuclear Properties for Even Zirconium(A=92-104) Isotopes. Quest: Journal of Geometry, Mathematical and Quantum Physics (Quest) 2026, 3(3), 167-175.

Received: 14th Jun 2026
Revised: 30th Jun 2026
Accepted: 18th Jul 2026
Published: 30th Jul 2026

Abstract: This study presents a theoretical analysis of several nuclear properties using the interacting boson model for some even-even zirconium isotopes in the isotope mass range $92 \le A \le 104$. Based on available experimental data, the Hamiltonians for each isotope were estimated within a theoretical framework based on the limit established by the ratio of the energy eigenvalues corresponding to the first and fourth excited levels to those corresponding to the first and second ones. From this perspective, an electric transition probability is an essential nuclear feature since it provides an indication of the degree of nucleus distortion. The published experimental results were partly consistent with the revised values for the lower energy states.

Keywords: Interacting Boson Model (IBM-1), Even-even zirconium isotopes, Nuclear structure, Energy levels, Electric quadrupole transition probability B(E2), Hamiltonian, Collective excitations, Nuclear deformation, Zirconium isotopes (A = 92–104), Algebraic nuclear model

1. Introduction

The Pauli exclusion principle governs the distribution of atomic electrons in particular orbitals. Electrons might detach entirely from the atom or travel to vacant (excited) orbits [1]. However, it has been demonstrated that the two comparable particle types that make up the nucleus—protons and neutrons are distributed independently of one another within energy levels subject to the limitations of the Pauli exclusion principle [2].

The atomic nucleus is not a uniform entity but rather encompasses a wide variety of nuclei, extending from hydrogen to the actinide elements. This diversity gives rise to a remarkable range of nuclear properties and phenomena. Despite its extremely small size, with a diameter on the order of 10^{-12} to 10^{-13}, the nucleus may contain up to several hundred protons and neutrons. These nucleons move relative to one another within the nucleus and interact primarily through the strong nuclear force and the electrostatic (Coulomb) force. [3].

The Interacting Boson Model (IBM) is an advanced algebraic approach developed as an alternative framework for describing collective nuclear states. In its original formulation, the model considers only two types of collective bosons with angular momenta (J = 0) (s-bosons) and (J = 2) (d-bosons). This simplified version, which treats protons and neutrons without distinction, is known as the Interacting Boson Model-1 (IBM-1) [4]. A boson interacts with the inert core of the nucleus (having closed shells) resulting in a single boson energy ε. Interaction of three-boson is ruled out by analogy with the assumptions of the standard shell model. Unlike the collective model, the IBM one does not obtain a semiclassical, vivid picture of the nucleus but rather describes the algebraic structure of the Hamiltonian operator and the states, and is therefore called an algebraic model [5].

The IBM-1 Hamiltonian can be expressed as [6]:

$$H^{\wedge} = \varepsilon_S (s^{\dagger} \cdot \tilde{s}) + \varepsilon_d (d^{\dagger} \cdot \tilde{d}) + \sum_{L=0,2,4} \tfrac{1}{2}(2L+1)^{1/2} C_L [[d^{\dagger} \times d^{\dagger}]^{(L)} \times [\tilde{d} \times \tilde{d}]^{(L)}]^{(0)} + 1/\sqrt{2}\, v_2 [[d^{\dagger} \times d^{\dagger}]^{(2)} \times [\tilde{d} \times \tilde{s}]^{(2)} + [d^{\dagger} \times s^{\dagger}]^{(2)} \times [\tilde{d} \times \tilde{d}]^{(2)}]^{(0)} + \tfrac{1}{2} v_0 [[d^{\dagger} \times d^{\dagger}]^{(0)} \times [\tilde{s} \times \tilde{s}]^{(0)} + [s^{\dagger} \times s^{\dagger}]^{(0)} \times [\tilde{d} \times \tilde{d}]^{(0)}]^{(0)} + \tfrac{1}{2} u_0 [[s^{\dagger} \times s^{\dagger}]^{(0)} + [\tilde{s} \times \tilde{s}]^{(0)}]^{(0)} + u_2 [[d^{\dagger} \times s^{\dagger}]^{(2)} \times [\tilde{d} \times \tilde{s}]^{(2)}]^{(0)} \quad (1)$$

where ($s^{\dagger}$, $\tilde{s}$) and ($d^{\dagger}$, $\tilde{d}$) are the operators for creating and annihilating (s) and (d) bosons, respectively. Two of the nine Hamiltonian parameters are one-body terms (ε_s , ε_d) seven two-body terms, [C_L (L = 0,2,4), v (L = 0,2), and u_L (L = 0,2)].(ε_s) and (ε_d) are the (C_L), (v_L), and (u_L) have been used to describe the energies of single bosons and their interactions in pairs. This indicates that when the number of bosons (N) is fixed, one one-body term and five two-body words are independent. This can be observed by considering ($N = ns + nd$).

Dynamical Symmetries

There are three limits in this model. The first determinations presented by (Arima and Iachello) in which the energy of the boson ε is much greater than the interaction potential. The Hamiltonian for vibration limit U(5) according to the following equation [7]:

$$\hat{H} = E\hat{n}_d + a_1\hat{L}.\hat{L} + a_3\hat{T}_3.\hat{T}_3 + a_4\hat{T}_4.\hat{T}_4 \quad (2)$$

where the parameters it contains are E, a_1, a_3 and a_4 only

The second one is rotational limit SU(3), the Hamiltonian function for this determination is given by [8]:

$$\hat{H} = a_1 L^{\wedge}.L^{\wedge} + a_2 Q^{\wedge}.\,Q^{\wedge} \quad (3)$$

γ-unstable limit of the IBM-1is $O(6)$ [9]:

$$\hat{H} = a_0\hat{P}.\hat{P} + a_1\hat{L}.\hat{L} + a_3\hat{T}_3.\hat{T}_3 \quad (4)$$

The Hamiltonian for transition region between SU(3) and O(6) [10]:

$$\hat{H} = a_0\hat{P}.\hat{P} + a_1\hat{L}.\hat{L} + a_2\hat{Q}.\hat{Q} \quad (5)$$

2. Materials and Methods

In addition to reproducing the excitation energy spectra of atomic nuclei, the Interacting Boson Model (IBM) provides an accurate description of electromagnetic transition rates. To achieve this, the transition operators are formulated in terms of boson creation and annihilation operators. In the IBM-1 framework, these operators are generally assumed to consist only of one-body terms involving the s and d bosons. Consequently, the model permits only transitions that can be described through one-body operators constructed from these two types of bosons. Therefore, such transitions constitute the complete set of electromagnetic transitions allowed within the IBM-1 formalism. The general equation for the probability of electromagnetic transition is written as follows:

$$\hat{T}^l_m = \alpha_2\delta_{l_2}\left[d^{\dagger} \times \tilde{s} + s^{\dagger} \times \tilde{d}\right]^2_m + \beta_l\left[d^{\dagger} \times \tilde{d}\right]^l_m + \gamma_0\delta_{l_0}\delta_{m_0}[s^{\dagger} \times \tilde{s}]^0_0 \quad (6)$$

where : l=0,1,2,3,4,…,; m=0,1,2,3,4,…, and $\gamma_{0,}\alpha_2,\beta_l$ represent the specific form of the transition operator, which are parameters specifying the various terms in the corresponding operators.

The reduced transition probability for electric is given by the following expression:

$$B(L, I_i \to I_f) = \frac{1}{2I_i+1}\left|< I_i\|\hat{T}^L\|I_f >\right|^2 \quad (7)$$

where $|< I_i\|\hat{T}^L\|I_f >|$ is the matrix element of (E2) transition.

From $B(E2)$ value, the electric quadruple moments of the nuclei can be derived as with I=

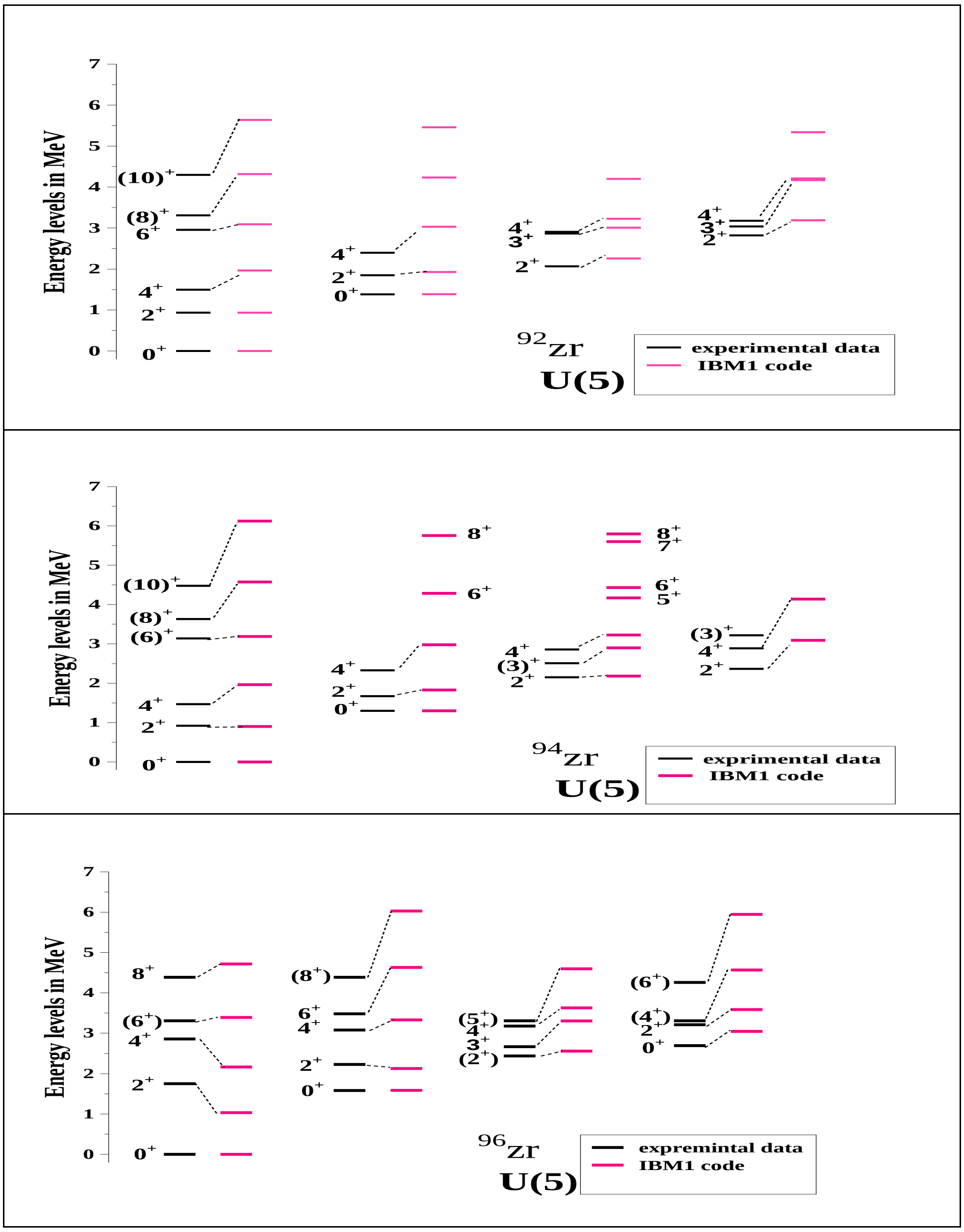
initial angular momentum:

$$Q_I = \sqrt{\frac{16\pi \times I(2I-1) \times B(E2)_{up}}{5(2I+1)(I+1)(2I+3)}} \qquad (8)$$

3. Results and Discussion

The study of nuclear structure involves extracting many nuclear properties, including the study of permitted energy levels, electrical transitions between these levels, the electric quadrupole moment, and other properties. The Hamiltonian coefficients for each isotope in this research were found by first estimating from the energy ratios of the experimental levels in data sheet as: ${E_{4_1}}/{E_{2_1}}$, ${E_{6_1}}/{E_{2_1}}$ and ${E_{8_1}}/{E_{2_1}}$.

Figure 1 include the energy levels of the isotopes ^{92}Zr to ^{104}Zr, including both experimental and theoretical results which applying IBM1 program, which were calculated by introducing the Hamiltonian coefficients and according to the specification for $^{92\text{-}104}$Zr [11].



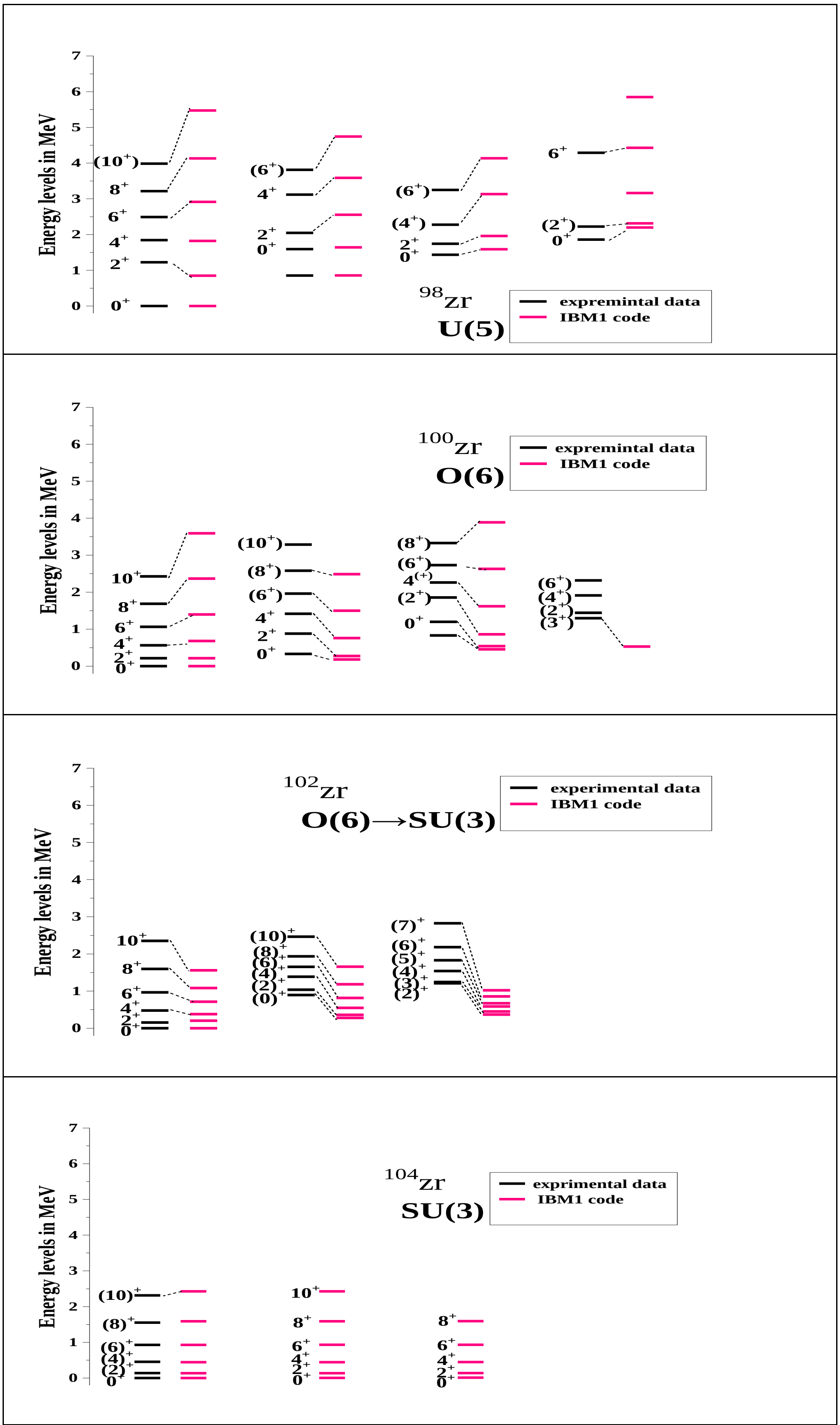


Figure (1): Energy levels from IBM1 program and the experimental data for all isotopes

The probabilities of electrical transitions for the isotopes in this study are listed in Tables 1-7, which include the values of some experimentally permitted transitions and compare them with theoretical results from the IBMT program. These transitions are a means of identifying the isotope type and the energy value released during the transition [12, 13].

Identifying the best isotope that gives off radiation suitable for medical uses without significant tissue damage or in nuclear applications as a nuclear fuel for unstable isotopes [14, 15].

Table (1). The electric transitions B(E2)↓ and $Q(2_1^+)$ for ^{92}zr

$I_i^+ \rightarrow I_f^+$	**B(E2) ↓ in e^2b^2**	
	Experimental data	**IBM-1 program**
$2_1^+ \rightarrow 0_1^+$	0.015796813±0.001481	0.00154790
$0_2^+ \rightarrow 2_1^+$	0.035542829±0.001234126	0.000258
$2_2^+ \rightarrow 0_2^+$	--	0.0010320
$2_3^+ \rightarrow 0_2^+$	--	0.0014447
$2_1^+ \rightarrow 2_2^+$	--	0.0025799
$3_1^+ \rightarrow 2_2^+$	--	0.0022113
$3_1^+ \rightarrow 2_3^+$	0.0059238±0.0034556	0.0015796
$4_1^+ \rightarrow 2_1^+$	0.009996421±0.0002962	0.0025799
$4_1^+ \rightarrow 2_2^+$	--	0.0004212
$4_2^+ \rightarrow 2_2^+$	--	0.0016216
$4_3^+ \rightarrow 2_3^+$	--	0.0019902
$6_1^+ \rightarrow 4_1^+$	≥2.41889E-06	0.0030959
$6_2^+ \rightarrow 4_2^+$	--	0.0021108
$6_3^+ \rightarrow 4_3^+$	--	0.0018919
$6_2^+ \rightarrow 4_3^+$	--	0.0012286
$Q\ (2_1^+ \rightarrow 2_1^+)$	--	-0.0803128 (eb)

Table (2). The electric transitions B(E2)↓ and $Q(2_1^+)$ for ^{94}zr

$I_i^+ \rightarrow I_f^+$	**B(E2) ↓e^2b^2**	
	Experimental data	**IBM-1 program**
$2_1^+ \rightarrow 0_1^+$	0.0124463±0.000762	0.0124458
$0_2^+ \rightarrow 2_1^+$	0.0238765±0.001016	0.00085342
$2_2^+ \rightarrow 0_2^+$	--	0.0006970
$2_3^+ \rightarrow 0_2^+$	--	0.0124458
$2_1^+ \rightarrow 2_2^+$	--	0.0213357
$3_1^+ \rightarrow 2_2^+$	--	0.0190497
$3_1^+ \rightarrow 2_3^+$		0.0010668
$4_1^+ \rightarrow 2_1^+$	0.0022327±5.84E-05	0.0213357
$4_1^+ \rightarrow 2_2^+$	--	0.0002845
$4_2^+ \rightarrow 2_2^+$	--	0.0139698
$4_3^+ \rightarrow 2_3^+$	--	0.0182877
$6_1^+ \rightarrow 4_1^+$		0.0266696
$6_2^+ \rightarrow 4_2^+$	--	0.0193961
$6_3^+ \rightarrow 4_3^+$	--	0.0195577
$6_2^+ \rightarrow 4_3^+$	--	0.0008297
$Q\ (2_1^+ \rightarrow 2_1^+)$	--	-0.0659998 (eb)

Table (3). The electric transitions B(E2)↓ and $Q(2_1^+)$ for ^{96}zr

$I_i^+ \to I_f^+$	B(E2) ↓e^2b^2	
	Experimental data	**IBM-1 program**
$2_1^+ \to 0_1^+$	0.006008442±0.00078371	0.0060083
$0_2^+ \to 2_1^+$	--	0.00042058
$2_2^+ \to 0_2^+$	>0.0070534	0.0002944
$2_3^+ \to 0_2^+$	--	0.0063087
$2_1^+ \to 2_2^+$	--	0.0105145
$3_1^+ \to 2_2^+$	--	0.0096562
$3_1^+ \to 2_3^+$	--	0.0004506
$4_1^+ \to 2_1^+$	--	0.0105145
$4_1^+ \to 2_2^+$	--	0.0001202
$4_2^+ \to 2_2^+$	--	0.0070812
$4_3^+ \to 2_3^+$	--	0.0096561
$6_1^+ \to 4_1^+$	--	0.0135186
$6_2^+ \to 4_2^+$	--	0.0102414
$6_3^+ \to 4_3^+$	--	0.0110152
$6_2^+ \to 4_3^+$	--	0.0003505
$Q\ (2_1^+ \to 2_1^+)$	--	-0.0428945 (eb)

Table (4). The electric transitions B(E2)↓ and $Q(2_1^+)$ for ^{98}zr

$I_i^+ \to I_f^+$	B(E2) ↓e^2b^2	
	Experimental data	**IBM-1 program**
$2_1^+ \to 0_1^+$	$0.00778703_{0.013425913}^{0.021481461}$	0.0778695
$0_2^+ \to 2_1^+$		0.00553738
$2_2^+ \to 0_2^+$	--	0.0847912
$2_3^+ \to 0_2^+$	--	0.0033917
$2_1^+ \to 2_2^+$	--	0
$3_1^+ \to 2_2^+$	--	0.0051913
$3_1^+ \to 2_3^+$		0.1297824
$4_1^+ \to 2_1^+$		0.1384346
$4_1^+ \to 2_2^+$	$0.144999863_{0.042962922}^{0.048333288}$	0.0346086
$4_2^+ \to 2_2^+$	--	0.1334905
$4_3^+ \to 2_3^+$	--	0.0951738
$6_1^+ \to 4_1^+$	--	0.1816954
$6_2^+ \to 4_2^+$	--	0.1586230
$6_3^+ \to 4_3^+$	--	0
$6_2^+ \to 4_3^+$	--	0
$Q\ (2_1^+ \to 2_1^+)$	--	-0.1455949 (eb)

Table (5). The electric transitions B(E2)↓ and $Q(2_1^+)$ for ^{100}zr

$I_i^+ \rightarrow I_f^+$	**B(E2) $\downarrow e^2b^2$**	
	Experimental data	**IBM-1 program**
$2_1^+ \rightarrow 0_1^+$	0.212404211±0.005516992	0.2124000
$0_2^+ \rightarrow 2_1^+$	0.184819248±0.016550977	0
$2_2^+ \rightarrow 0_2^+$	--	0.0647314
$2_3^+ \rightarrow 0_2^+$	--	0.1203600
$2_1^+ \rightarrow 2_2^+$	--	0.2925919
$3_1^+ \rightarrow 2_2^+$	--	0.2311838
$3_1^+ \rightarrow 2_3^+$	--	0.1074643
$4_1^+ \rightarrow 2_1^+$	--	0.2925918
$4_1^+ \rightarrow 2_2^+$	--	0
$4_2^+ \rightarrow 2_2^+$	--	0.1695347
$4_3^+ \rightarrow 2_3^+$	--	0
$6_1^+ \rightarrow 4_1^+$	$0.3586045_{0.063445414}^{0.096547369}$	0.3236571
$6_2^+ \rightarrow 4_2^+$	--	0.2238099
$6_3^+ \rightarrow 4_3^+$	$6.62\mathrm{E}-05_{0.019309474}^{0.027584962}$	0.1287237
$6_2^+ \rightarrow 4_3^+$	--	0
$Q\ (2_1^+ \rightarrow 2_1^+)$	--	-0.0000056 (eb)

Table (6). The electric transitions B(E2)↓ and $Q(2_1^+)$ for ^{102}zr

$I_i^+ \rightarrow I_f^+$	**B(E2) $\downarrow e^2b^2$**	
	Experimental data	**IBM-1 program**
$2_1^+ \rightarrow 0_1^+$	0.297391528±0.039652204	0.2964483
$0_2^+ \rightarrow 2_1^+$	--	0.0002354
$2_2^+ \rightarrow 0_2^+$	--	0.0221788
$2_3^+ \rightarrow 0_2^+$	--	0.1041625
$2_1^+ \rightarrow 2_2^+$	--	0.0319506
$3_1^+ \rightarrow 2_2^+$	--	0.4483263
$3_1^+ \rightarrow 2_3^+$	--	0.0703090
$4_1^+ \rightarrow 2_1^+$	--	0.3408976
$4_1^+ \rightarrow 2_2^+$	--	0.0493246
$4_2^+ \rightarrow 2_2^+$	--	0.0413560
$4_3^+ \rightarrow 2_3^+$	--	0.0734458
$6_1^+ \rightarrow 4_1^+$	--	0.4759817
$6_2^+ \rightarrow 4_2^+$	--	0.0465586
$6_3^+ \rightarrow 4_3^+$	--	0.0942232
$6_2^+ \rightarrow 4_3^+$	--	0.0211602
$Q\ (2_1^+ \rightarrow 2_1^+)$	--	-1.2951550 (eb)

Table (7). The electric transitions B(E2)↓ and $Q(2_1^+)$ for ^{104}zr

$I_i^+ \rightarrow I_f^+$	**B(E2) $\downarrow e^2b^2$**	
	Experimental data	**IBM-1 program**
$2_1^+ \rightarrow 0_1^+$	0.5231859±0.0087198	0.5260672

$0_2^+ \rightarrow 2_1^+$	--	0.01062412
$2_2^+ \rightarrow 0_2^+$	--	0.2307754
$2_3^+ \rightarrow 0_2^+$	--	0.3253897
$2_1^+ \rightarrow 2_2^+$	--	0.8995272
$3_1^+ \rightarrow 2_2^+$	--	0.5830317
$3_1^+ \rightarrow 2_3^+$	--	0.4297923
$4_1^+ \rightarrow 2_1^+$	--	0.3827711
$4_1^+ \rightarrow 2_2^+$	--	0.0579717
$4_2^+ \rightarrow 2_2^+$	--	0.1215304
$4_3^+ \rightarrow 2_3^+$	--	0.1545352
$6_1^+ \rightarrow 4_1^+$	--	0.5457680
$6_2^+ \rightarrow 4_2^+$	--	0.1609635
$6_3^+ \rightarrow 4_3^+$	--	0.0228469
$6_2^+ \rightarrow 4_3^+$	--	0.1268303
$Q\ (2_1^+ \rightarrow 2_1^+)$	--	-0.7368304 (eb)

4. Conclusion

The group of zirconium isotopes studied in this research is a series of nuclei that exhibit properties extending from the vibrational specificity of the first four nuclei Zr. Zr having O(6) properties, then ^{102}Zr taking the properties of a transitional specificity between vibrational and gamma unstable, then moving to a rotational specificity, forming the final nucleus Zr with SU(3) properties. Many levels not confirm in the isotopes were been found. The last isotope, all of its levels are experimentally unconfirmed. Through theoretical results, it was confirmed, and several other levels were found. The correspondence between experimental values and theoretical onedemonstrates the effectiveness of the program in studying the nuclear properties of the nuclei of the isotopes in this research. The electrical transitions of the studied group of isotopes are a measure of the probability of transitions between levels, in addition to calculated value of the electric quadrupole moment, which indicates the charge distribution within the nucleus.

REFERENCES

[1] H. A. Bethe and R. F. Bacher, "Nuclear physics A. Stationary states of nuclei", Reviews of Modern Physics, 8(2), 82, (1936).

[2] J. M.Blatt and V. F. Weisskopf, "Theoretical nuclear physics ", Courier Corporation, (1991).

[3] D. J. Rowe, "Nuclear collective motion: models and theory" ,World Scientific, (2010).

[4] P. Van Isacker, "Symmetries of the interacting boson model", Frontiers of Physics, 13(6), 132107, (2018)

[5] A. Arima and F. Iachello, "The Interacting Boson Model", Cambridge University Press, Cambridge, UK, (1987).

[6] K. Abrahams., K. Allaart and A.E.L. Dieperink, *"Nuclear Structure"*, Plenum press, New York and London (1981).

[7] H. A. Marhoon, " *Study of the energy levels for even-even $^{98\text{-}104}$Mo by using IBM-1*", M.Sc. Thesis, University of Babylon, (2010).

[8] H.G. Borner, W.F. Davidson, J. A. Pinston and S. Bachk, J. Nuclear Phys. A, Vol. 297, p. (373), (1978).

[9] D. Bondatsos and E. David, *"Interacting Boson Models of Nuclear Structure"* , Stanford, Pub. In the United State, Oxford University Press, New York, (1988).

[10] J.K. Tuli, Nuclear Data Sheets, Vol. 11, No.1, p. (12), (1998).

[11] A. Sethi, N. M. Hintz, D. N. Mihailidis, A. M. Mack and M. Gazzaly, Phys. Rev. C, Vol. 44, No.2, p. (700), (1991).

[12] H. A. Marhoon, " *Study of the energy levels for even-even $^{98-104}$Mo by using IBM-1*", M.Sc. Thesis, Babylon University, (2010).
[13] H. J. Arnikar, Essentials of Nuclear Chemistry, Fourth rev. New age International L.td., 2011.
[14] Gh. A. Jaber and M. K. Muttaleb, *"Studying the breaking symmetry for O (6) even Hg isotopes in interacting boson model (1 and 2)*", AIP Conference Proceedings, 2144(1), 030014, (2019).
[15] M. K. Saydullayevich, "The Role of Leadership in Management Systems and Its Impact on Organizational Effectiveness: Evidence from Uzbekistan and Central Asian Enterprises," *Central Asian Journal of Innovations on Tourism Management and Finance*, vol. 7, no. 3, pp. 226–232, 2026.