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Relationship Between Non-Wandering Point and Stability in Random Dynamical Systems

Akhlas Kadom Obaid[1]
Ministry of Education, Open College of Education, Najaf center, Iraq
*Correspondence: ikhlas.kadim.eubaid@ec.edu.iq

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Abstract: This paper aims to study the concepts of random dynamical systems (RDSs) and investigates the relationships between several fundamental concepts, including recurrent points, Poisson stability, Lyapunov stability, Lipschitz stability, and non-wandering points. The main results show that any positively or negatively Poisson stable point is necessarily a non-wandering point and that recurrent points in Lyapunov or orbitally Lipschitz stable systems are also non-wandering. In addition, it is proven that the set of Poisson stable points is dense in a complete metric space in which every point is non-wandering. Overall, the study enhances the understanding of the interplay between stability and recurrence in random dynamical systems and clarifies their qualitative behavior in stochastic settings.

Keywords: Random Dynamical Systems, recursive of random dynamical systems, Poisson, Lyapunov stable, Lipschitz stable,and non-wandering point of random dynamical.

1. Introduction

The theory of dynamical systems originated from the pioneering work of Henri Poincaré, whose studies of recurrence and qualitative behavior laid the foundations of modern dynamics. During the twentieth century, growing interest in the effects of randomness on natural and engineered systems led to the development of stochastic and random dynamical models. A significant advancement in this field was achieved by Ludwig Arnold, who established a rigorous framework for Random Dynamical Systems (RDSs), integrating ideas from probability theory, ergodic theory, and dynamical systems theory. Since then, considerable attention has been devoted to studying invariant sets, attractors, recurrence, and stability in random environments [1,2].

This paper investigates the relationship between non-wandering points see [3,4] and several stability concepts in random dynamical systems [5,6]. We examine the connections among Poisson stability, recurrence, Lyapunov stability, and Lipschitz stability, and establish results that clarify their interactions. In addition, we provide conditions under which recurrent and Poisson stable points are non-wandering. These results contribute to a better understanding of the qualitative behavior of random dynamical systems and the interplay between stability and recurrence.

2. Some concepts on RDSs:

Definition 2.1 [5,7]:
The $(\mathbb{T},\Omega,\mathcal{F},\mathbb{P},\vartheta)$ is said to be ***metric dynamical system*** (Shortly MDS) if $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space and the function $\vartheta:\mathbb{T}\times\Omega\to\Omega$ satisfy
(i) ϑ is measurable,
(ii) $\vartheta(0,\omega)=\omega$, for evry $\omega\in\Omega$,
(iii) $\vartheta_{t+s}(\omega)=(\vartheta_t\circ\vartheta_s)(\omega)$ for evry $\omega\in\Omega$, $t,s\in\mathbb{T}$ and
(iv) $\mathbb{P}(\vartheta_t F)=\mathbb{P}(F)$, $\forall\, F\in\mathcal{F}$ and $\forall\, t\in\mathbb{T}$.

Definition 2.2 [5,8] (Random Dynamical System): Let X be a locally compact Hausdorff space and let $(\Omega,\mathcal{F},\mathbb{P},\vartheta)$ be a Metric Dynamical System (MDS) . A topological random dynamical system (RDS) on X over ϑ is a mapping $\varphi:\mathbb{T}\times\Omega\times X\to X$, which satisfies the following conditions:
(i) Continuity:
for every $\omega\in\Omega$, the mapping $\varphi(\cdot,\omega,\cdot):\mathbb{T}\times X\to X$ is continuous.
(ii) Cocycle Property:
The mapping $\varphi(t,\omega):=\varphi(t,\omega,\cdot):X\to X$ form a cocycle over $\vartheta(\cdot)$, that is satisfy
$\varphi(0,\omega)x=x \quad \forall\omega\in\Omega \;,\; x\in X,$ and
$\varphi(t+s,\omega)=\varphi(t,\vartheta_s\omega)\circ\varphi(s,\omega) \;,\forall\; s,t\in\mathbb{T},\omega\in\Omega.$

Definition 2.3 [2]:
Suppose that (X,d) be a metric space which is a measurable space with Borel σ −field $\mathcal{B}(X)$ and $(\Omega,\mathcal{F})$ be a measurable space. The set-valued function $A:\Omega\to\mathcal{B}(X)$, $\omega\mapsto A(\omega)$, is a **random set** if the mapping $\omega\mapsto d(x,A(\omega))$ is measurable for each $x\in X$. The random set $A(\omega)$ is called a ***random closed(compact) set***, if it is closed (compact) for all $\omega\in\Omega$.

Definition 2.4 [2]:
Let $(\mathbb{T},\Omega,\mathcal{F},\mathbb{P},\vartheta)$ be a MDS. A random variable $\delta:\Omega\longrightarrow\mathbb{R}^+$ is called a ***tempered random variable (t.r.v.)***, if there exists a measurable subset $\widetilde{\Omega}$ of Ω such that $\lim\limits_{n\to\pm\infty}\frac{1}{|n|}\log^+\delta(\vartheta_n\omega)=0$, $\forall\omega\in\widetilde{\Omega}$, where $\log^+(x):=\max\{0,\log\}$.

Definition 2.5[9,10]:
Let (φ,ϑ) be a random dynamical system . For a given point $x\in X$, define the set-valued mapping
$\gamma_x^t:\omega\to D(\omega)$ by $\gamma_x^t(\omega)=\cup_{t\in T}\{\varphi(t,\vartheta_{-t}\omega)x\}$
the set γ_x^t is called a ***trajectory(orbit)*** of the point x .

Definition 2.6[9,10]: Suppose that (φ,ϑ) is a LRDS, and $A\subseteq X$ is a nonempty random set.
(a) A is called a **minimal random set** if satisfies the following:
1) A is an invariant.
2) A is a closed .
3) no proper subset of A has their properties.
(**b**) If X itself is a minimal set , then we call (ϑ,φ) a **minimal *RDS***.
(**c**) $x\in X^\Omega$ is called a **minimal point** if $\overline{\gamma_x(\omega)}$ is a minimal random set.
If every random variable $x\in X^\Omega$ is minimal point, then (ϑ,φ) is called ***pointwise minimal***.
See[11,12]

3. some types of stability and non-wandering point :

Definition 3.1[13]: Let M be a random set. **The P−omega limit** set of M , denoted by $\Gamma_M^P(\omega)$, is defined by

$$\Gamma_M^P(\omega)=\{y\in X:\exists\, net\,\{t_\lambda\}\in P, t_\lambda\longrightarrow\infty,\{x_\lambda\}\in M(\vartheta-t_\lambda\omega)\ni \psi(t_\lambda,\vartheta-t_\lambda\omega)x_\lambda\longrightarrow y\}$$

Definition 3.2[13]: Let M be a random set. The ***P−prolongational limit set of*** M , denoted by $J_M^P(\omega)$, is defined by

$$J_M^P(\omega) := \{y \in X: \exists\ nets\ \{x_\lambda\}, \{t_\lambda\}, t_\lambda \longrightarrow +\infty \ \ni\ d(x_\lambda, M(\vartheta - t_\lambda\omega)) \longrightarrow 0\ , and$$
$$\varphi(t_\lambda, \vartheta - t_\lambda\omega)x_\lambda \longrightarrow y\}$$

Definition 3.3[13]: Let M be a random set. The **P−prolongation of a random set** M , denoted by $D_M^P(\omega)$, is defined by

$$D_M^P(\omega) := \{y \in X: \exists\ nets\ \{x_\lambda\}, \{t_\lambda\} \ni\ d(x_\lambda, M(\vartheta - t_\lambda\omega)) \to 0\ ,$$
$$\varphi(t_\lambda, \vartheta - t_\lambda\omega)x_\lambda \to y\ \}$$

Theorem 3.4[13] : For every t in T and ω∈Ω. Then $y \in J_x^P(\omega)$ if and only if $x \in J_y^{P^{-1}}(\omega), where\ y = \varphi(t, \vartheta_{-t}\omega)x.$

Proof. Let $y \in J_x^P(\omega)$. So there are two nets $\{t_\lambda\}$ ∈ P and $\{x_\lambda\}$ ∈ X such that $x_\lambda \longrightarrow x$, $t_\lambda \longrightarrow +\infty$ and $\varphi(t_\lambda, \vartheta_{-t_\lambda}\omega)x_\lambda \longrightarrow y.$ Set $t_\lambda := -t_\lambda\ and\ y_\lambda := \varphi(t_\lambda, \vartheta_{-t_\lambda}\omega)x_\lambda$. Then $\{\tau_\lambda\}$ is a net ∈ P^{-1} , $\tau_\lambda \longrightarrow -\infty$ and $\{y_\lambda\}$ is a net ∈ X , $y_\lambda \longrightarrow y$. Now

$$d(\varphi(\tau_\lambda, \vartheta_{-\tau_\lambda}\omega)y_\lambda, x) = d(\varphi(\tau_\lambda, \vartheta_{-\tau_\lambda}\omega) \circ \varphi(t_\lambda, \vartheta_{-t_\lambda}\omega)x_\lambda, x)$$
$$= d(\varphi(\tau_\lambda, \vartheta_{-\tau_\lambda}\omega) \circ \varphi(t_\lambda, \vartheta_{-t_\lambda}\omega)x_\lambda, \varphi(\tau_\lambda, \vartheta_{-\tau_\lambda}\omega)y),$$
$$= d(\varphi(t_\lambda, \vartheta_{-t_\lambda}\omega)x_\lambda, y) \longrightarrow 0$$

Then we have $\varphi(\tau_\lambda, \vartheta_{-\tau_\lambda}\omega)y_\lambda \longrightarrow x$ .Thus $x \in J_y^{P^{-1}}(\omega)$.

Similarly we can prove the converse. ■

Definition 3.5[14]: A random set $A \subset X$ will be said to be ***p- recursive*** with respect to a random set $B \subset X$ if for each cmpact $K \in T$ there is a $t \in P - K$ and an $x \in B(\omega)$ such that

$\mathbb{p} = \{\omega: \varphi(t, \vartheta_{-t}\omega)x \in A(\omega)\} = 1$

Or equivalenty .

$\mathbb{p} = \{\omega: \varphi(-t, \omega)A(\omega) \cap B(\omega) \neq \emptyset\} = 1$

A set A is called self P-recusive if

$\mathbb{p} = \{\omega: \varphi(-t, \omega)A(\omega) \cap A(\omega) \neq \emptyset\} = 1$

Definition 3.6 [14]: A point $x \in X$ is said to be ***non-wandering*** if every Random neighborhood U of x is self recursive.this mean that

$\mathbb{p} = \{\omega: \varphi(-t, \omega)U(\omega) \cap U(\omega) \neq \emptyset\} = 1$

Theorem 3.7 [14]: Let $x \in X$.Then the following statements are equivalent.

(a) x is a P−non-wandering point ,

(b) $x \in J_x^P(\omega)$,

(c) every random neighborhood of x is self P^{-1}− recursive,

(d) $x \in \Gamma_M^{P^{-1}}(\omega)$.

Proof. We prove the equivalence by showing the circular implications .

(a)⇒ (b): Assume that x is a P-non-wandering point . Consider $\{r_\lambda\}$ be a null net of positive random variables, $r_\lambda \longrightarrow 0$, and a sequence $\{t_\lambda\}$ in $\mathbb{R}$ with $t_\lambda \longrightarrow +\infty$. Since each random neighborhood $S(x, r_\lambda(\omega))$ is positively recursive , there exist elements $x_\lambda \in S(x, r_\lambda(\omega))$ and a $\tau_\lambda > t_\lambda$ with $\varphi(t_\lambda, \vartheta_{-t_\lambda}\omega)x_\lambda \in S(x, r_\lambda(\omega))$. Because $r_\lambda \longrightarrow 0$ we obtain $x_\lambda \longrightarrow x$ and $\varphi(\tau_\lambda, \vartheta_{-t_\lambda}\omega)x_\lambda \longrightarrow x$ and since $\tau_\lambda \longrightarrow +\infty$ we conclude $x \in J_x^+(\omega)$. Thus (b) holds.

(a)⟹(c) Now suppose (a). Then there exists a net $\{x_\lambda\}$ in X and a net $\{t_\lambda\} \subset \mathbb{T}$ with $x_\lambda \longrightarrow x$ and $t_\lambda \longrightarrow +\infty$ such that $\varphi(t_\lambda, \vartheta_{-t_\lambda}\omega)\, x_\lambda \longrightarrow x$. Now for any random neighborhood $U(\omega)$ of x the net x_λ is eventually in $U(\omega)$ for every ω and $\varphi(t_\lambda, \vartheta_{-t_\lambda}\omega)\, x_\lambda$ eventually in $U(\omega)$ for every ω. Moreover, since also

$$\varphi(t_\lambda, \vartheta_{-t_\lambda}\omega)x_\lambda \to x,$$

it follows that this image net is eventually contained in $U(\omega)$ as well. Therefore, every random neighborhood of x is self P^{-1} −recursive, and (c) holds.

So the point x is non-wandering .

(c) if and only if (d). Is proved in the same way.

(b) if and only if (d): by Theorem (3.4). ■

Theorem 3.8[14] : Let $A \subset X$. every limit point x of A is P−non-wandering whenever either x

$\in \Gamma_x^P\ (\omega)$ or $x \in \Gamma_x^{P^{-1}}(\omega)$.

Proof: Let $\{x_\lambda\}$ be a net in A with $x_\lambda \longrightarrow x$. To show that $x \in J_x^P(\omega)$. In fact for each , either $x_\lambda \in \Gamma_{x_\lambda}^P(\omega)\ or\ x_\lambda \in \Gamma_{x_\lambda}^{P^{-1}}(\omega)$.

Assume $x_\lambda \in \Gamma_{x_\lambda}^P(\omega)$ for all λ. Then

$$d(x_\lambda, \varphi(t_\lambda, \vartheta - t_\lambda\omega)x_\lambda) \longrightarrow 0.$$

Then clearly

$$d(x, \varphi(t_\lambda, \vartheta - \mathrm{t}\lambda\,\omega)x_\lambda) < d(x, x_\lambda) + d(x_\lambda, \varphi(t_\lambda, \vartheta_{-t_\lambda}\omega)x_\lambda) \longrightarrow 0.$$

This shows that $\varphi(t\lambda, \vartheta_{-t_\lambda}\omega)x_\lambda \longrightarrow x$ and so $x \in J_x^P(\omega)$. For the second situation, similar considerations lead to the desired conclusion $x \in J_x^{P^{-1}}(\omega)$. By definition 3.1 we get our result.

Definition 3.9: A point $x \in X$ is said to be positively Poisson stable if every random neighborhood of x is recursive with respect to $\{x\}$.

Definition 3.10. A point $x \in X$ is Poisson stable whenever, $x \in \Gamma_M^P\ (x)$. If a point $x \in X$ is Poisson stable then both $\varphi(t,\omega)x$ and the trajectory $\gamma_x^t(\omega)$ are said to be Poisson stable.

Definition 3.11[10, 15]: An RDS (ϑ, φ) on X is said to be **orbitally (positively) Lipschitz stable at $x \in X$** if there exist a random variable $K_x: \Omega \longrightarrow [1, \infty)$ and a constant $\delta = \delta(x) > 0$ such that

$\|\varphi(t, \vartheta_{-t}\omega)a(\vartheta_{-t}\omega) - \varphi(t, \vartheta_{-t}\omega)b(\vartheta_{-t}\omega)\| \leq K(\omega)\|a(\omega) - b(\omega)\|$,

$\forall\ t \in \{\mathbb{R}^+\}\ \mathbb{R}$ and $\forall\ a, b \in \gamma_x(\omega)$ with $\|a(\omega) - b(\omega)\| < \delta$.

The system (ϑ, φ) is said to be orbitally (positively) Lipschitz stable if it is orbitally (positively) Lipschitz stable at every point $x \in X$.

Definition 3.12[10,15]: A random dynamical system(ϑ, φ) on X is said to be **orbitally (positively) Lyapunov stable at** a point $x \in X$ if for any TRV $\varepsilon(\omega) > 0$ there exists a constant poistive δ such that

$$\|\varphi(t, \vartheta_{-t}\omega)a(\vartheta_{-t}\omega) - \varphi(t, \vartheta_{-t}\omega)b(\vartheta_{-t}\omega)\| \leq \varepsilon(\omega)$$

$\forall\ t \in \{\mathbb{R}^+\}\ \mathbb{R}$ and all $a, b \in \gamma_x(\omega)$ with $\|a(\vartheta_{-t}\omega) - b(\vartheta_{-t}\omega)\| < \delta(\omega)$.

The system (ϑ, φ) is orbitally (positively) Lyapunov stable provided that every point x in X is orbitally (positively) Lyapunov stable.

Proposition 3.13[10]: is orbitally (positively) Lipschitz stable at $x \in X$ implying (ϑ, φ) is orbitally (positively) Lyapunov stable at x. Hence, orbitally (positively) Lipschitz stable dynamical systems must be orbitally (positively) Lyapunov stable dynamical systems. Orbitally (positively) Lipschitz stability of (ϑ, φ) at $x \in X$ implies its orbitally (positively) Lyapunov stability at the same point. Hence, any orbitally (positively) Lipschitz stable dynamical system must also be orbitally (positively) Lyapunov stable.

Proof. Assume that (ϑ, φ) is orbitally (positively) Lipschitz stable at $x \in X$ then there exist a tempered random variable TRV $K(\omega) \geq 1$ and a constant $\delta = \delta(\omega) > 0$ such that

$$\begin{aligned}\|\varphi(t, \vartheta_{-t}\omega)a(\vartheta_{-t}\omega) - \varphi(t, \vartheta_{-t}\omega)b(\vartheta_{-t}\omega)\| &\leq K(\omega)\|a(\omega) - b(\omega)\| \\ &\leq \delta(\omega)\end{aligned}$$

for all $t \in \{\mathbb{R}^+\}\ \mathbb{R}$ and all $a, b \in \gamma_x(\omega)$ satisfying $\|a(\omega) - b(\omega)\| < \delta(\omega)$.

Now choosing $\varepsilon(\omega) = \delta(\omega) > 0$, we obtain

$$\|\varphi(t, \vartheta_{-t}\omega)a(\vartheta_{-t}\omega) - \varphi(t, \vartheta_{-t}\omega)b(\vartheta_{-t}\omega)\| \leq \varepsilon(\omega)$$

$for\ every\ \ t \in \{\mathbb{R}^+\}\ \mathbb{R}$ and all $a, b \in \gamma_x(\omega)$ with $\|a(\omega) - b(\omega)\| < \delta(\omega)$. ■

Note:Suppose that $A(\omega)$ and $B(\omega)$ are two nonempty random sets in X satisfying $A(\omega) \subseteq \overline{B(\omega)}$.

Definition 3.14[10]: Let (ϑ, φ) be a random dynamcal systems (RDS) . The trajectory $\gamma_p(\omega)$ is said to ***uniformly approximates*** a random set Q , with $p \in Q(\omega)$, provied that for every TRV(ε), there exists a positive constant $L = L(\varepsilon)$ such that for every for all $t_0 \in \mathbb{T}$ we have $Q(\omega) \subseteq B(\gamma_p^I(\omega), \varepsilon)$.

Definition 3.15[10]: An random dynamical system (RDS) (ϑ, φ) is said to be **recurrent** if , for every point $p \in X$ the trajectory $\gamma_p^t(\omega)$ uniformly approximates itself **.**

Lemma 3.16[10]: Every trajectory in a compact minimal set is said to be recurrent. Consequently, the closure of a recurrent trajectory is obtained as the whole compact minimal set.

Proof. Let $M(\omega)$ be a compact minimal random set. Assume that there exists a point $x \in M(\omega)$ such that (ϑ, φ) is not recurrent at x. Then, there exists a random variable TRV $\varepsilon(\omega) > 0$ and sequences $\{T_n\}, \{t_n\}, \{\tau_n\}$, with $T_n > 0, T_n \to +\infty$, and

$$\varphi(\tau_n, \vartheta_{-\tau_n}\omega)x \notin B\big(\varphi(l, \vartheta_{-l}\omega)x, \varepsilon(\omega)\big), n = 1, 2, \ldots$$

where, $l = [t_n - T_n, t_n + T_n]$. This shows that

$$\|\varphi(\tau_n, \vartheta_{-\tau_n}\omega)x - \varphi(t_n, \vartheta_{-t_n}\omega) \circ \varphi(t, \vartheta_{-t}\omega)x\| \geq \varepsilon(\omega)$$

Whenever $|t| \leq T_n, n = 1,2, \ldots$. The sequences

$\{\varphi(\tau_n, \vartheta_{-\tau_n}\omega)x\}$, $\{\varphi(t_n, \vartheta_{-t_n}\omega)x\}$

are contained in the compact set M and

$\varphi(t_n, \vartheta_{-t_n}\omega)x \to y$, $\varphi(\tau_n, \vartheta_{-\tau_n}\omega)x \to z$.

Then $y, z \in \overline{\gamma_x(\omega)} = M$. Consider now (ϑ, φ) at y**.** For any $t \in \mathbb{R}$, the inequality

$$\|\varphi(t, \vartheta_{-t}\omega)y - z\| \geq \|\varphi(t_n, \vartheta_{-t_n}\omega) \circ \varphi(t, \vartheta_{-t}\omega)x - \varphi(\tau_n, \vartheta_{-\tau_n}\omega)x\|$$
$$-\|\varphi(t, \vartheta_{-t}\omega)y - \varphi(t_n, \vartheta_{-t_n}\omega) \circ \varphi(t, \vartheta_{-t}\omega)x\| - \|\varphi(\tau_n, \vartheta_{-\tau_n}\omega)x - z\|$$

yields

$$\|\varphi(t, \vartheta_{-t}\omega)y - z\| \geq \varepsilon$$

from the inequality. Thus $z \notin \overline{\gamma_y(\omega)}$ i.e. $z \notin M$ as $M = \overline{\gamma_y(\omega)}$

This contradiction. ■

Theorem 3.17[10]: Let (ϑ, φ) be an orbitally (positively) Lipschitz stable .Then x a point in X is recurrent if and only if it is negatively Poisson stable.

Proof. Let x be a negatively Poisson stable point in X. Then we will prove that $\gamma_x(\omega)$ is minimal. Suppose that $y \in \overline{\gamma_x(\omega)}$. Then it is enough to show that $\overline{\gamma_x(\omega)} \subseteq \overline{\gamma_y(\omega)}$since $\overline{\gamma_x(\omega)}$ is closed and invariant. By the negative Poisson stability of the point x, there exists a sequence $\{t_n\}$ in $\mathbb{R}^-$ such that $x_n = \varphi(t_n, \vartheta_{-t_n}\omega)x$ and $x_n \to y$, $t_n \to -\infty$.

Let $z \in \overline{\gamma_x(\omega)}$ then for any TRV $\varepsilon(\omega)$ there exists $r \in \mathbb{R}$ such that

$x \in B(\varphi(-r, \vartheta_r\omega)z, \varepsilon)$.

Let us choose $\delta_1 > 0$ such that

$$\overline{B(x, \delta_1)} \subseteq B(\varphi(-r, \vartheta_r\omega)z, \varepsilon).$$

Since the orbit $\gamma_x(\omega)$is positively Lipschitz stable, there are $\delta_2 > 0$ and $K(\omega) \geq 1$ such that

$$\|\varphi(t, \vartheta_{-t}\omega)a - \varphi(t, \vartheta_{-t}\omega)b\| \leq K(\omega)\|a - b\|,$$

for all $t \in \mathbb{R}^+$ and all $a, b \in \gamma_x(\omega)$ with $\|a - b\| < \delta_2$.

Put $\delta = \min\{\delta_1, \delta_2\}$.Since $\{x_n\}$ converges, there exists m such that $\|x_n - x_m\| < \delta/K$ for all $n \geq m$, and $x_m = \varphi(t_m, \vartheta_{-t_m}\omega)x$.

Hence we have

$$\|x - \varphi(t_n, \vartheta_{-t_n}\omega) \circ \varphi(t_m, \vartheta_{-t_m}\omega)x\| \leq K(\omega)\|x_n - x_m\| < \delta$$

if $n \geq m$. So, $x_n \in B(x_m, \delta)$, for each $t_n \leq t_m$. Consequently we get

$$y \in \overline{B(x_m, \delta)} \subseteq \overline{B(x, \delta)}t_m$$
$$\subseteq B\big(\varphi(-r, \vartheta_r\omega) \circ \varphi(t_m, \vartheta_{-t_m}\omega)z, \varepsilon\big)$$

Thus we obtain $B(z, \varepsilon) \cap \gamma_y(\omega) \neq \emptyset$. This proves that $\overline{\gamma_x(\omega)}$ is minimal. Also $\overline{\gamma_x(\omega)}$ is compact . As each orbit in a compact minimal set is recurrent, by Lemma (3.16) we have the point x is also recurrent. ■

Theorem 3.18: Assume that (ϑ, φ) is a random dynamical system.Let $P \subset X$ be such that for all $x \in P$is either positively or negatively Poisson stable. Then every $x \in \bar{P}$ is non-wandering.

Proof. Let $\{x_\lambda\}$ be a net in P and $x_\lambda \to x$. We need to show that $x \in J_x^+(\omega)$.

for each λ either we have $x_\lambda \in \Gamma_{x_\lambda}^p(\omega)$ or $x_\lambda \in \Gamma_{x_\lambda}^{p^{-1}}(\omega)$. Thus, by passing to a subsequence, we

may assume that either $x_\lambda \in \Gamma^{p}_{x_\lambda}(\omega)$ for all λ
or $x_\lambda \in \Gamma^{p^{-1}}_{x_\lambda}(\omega)$.

Assume $x_\lambda \in \Gamma^{p}_{x_\lambda}(\omega)$. For each λ there is a $t_\lambda > \lambda$ with $d(x_\lambda, \varphi(t_\lambda, \vartheta - t_\lambda\omega)x_\lambda) < 1/\lambda$

Then clearly

$$d(x, \varphi(t_\lambda, \vartheta - t\lambda\, \omega)x_\lambda) \le d(x, x_\lambda) + d\big(x_\lambda, \varphi\big(t_\lambda, \vartheta_{-t_\lambda}\omega\big)x_\lambda\big)$$
$$\le d(x, x_\lambda) + 1/\lambda$$

This shows that $\varphi\big(t_\lambda, \vartheta_{-t_\lambda}\omega\big)x_\lambda \to x$ and consequently
$x \in J^{+}_{x}(\omega)$. In the second case similar considerations show that $x \in J^{-}_{x}(\omega)$.
Thus by Theorem 2.12 every $x \in P$ is non-wandering. ■

Theorem 3.19: Let (ϑ, φ) is RDS and X be a complete. Let every $x \in X$ be non-wandering .Then the set of Poisson stable points P is dense in X.

Proof. Let $x \in X$ and $r(\omega) > 0$. Let $y \in S\,(x, r(\omega))$ such that
$y \in \Gamma^{+}_{y}(\omega)\ and\ y \in \Gamma^{-}_{y}(\omega). Let\ S\,(x, r(\omega)) = U.$ Since U is self positively recursive, there is a $t_1 > 1$ such that $U \cap U_{t_1} \neq \emptyset$.

And $U \cap U_{t_1}$is open. We take an $x_1 \in U \cap U_{t_1}$ and positive $r_1(\omega) < \frac{1}{2}$ such that
$S\,(x_1, r_1(\omega)) \subset U \cap U_{t_1}$. Set $U1 = S\,(x_1, r_1(\omega))$.

Since x_1 is non-wandering, U_1 is self negatively recursive, and there is a $t_2 > 2$ such that $U_1 \cap U_{1_{t_2}} \neq \emptyset$.

Take x_2 in $U_1 \cap U_{1_{t_2}}$and a positive $r_2(\omega) < {}^{1}\!/_{2^2}$ such that
$S\,\big(x_2, r_2(\omega)\big) \subset U_1 \cap U_{1_{t_2}}$. Set $U_2 = S\,(x_2, r_2(\omega))$.

Since x_2 is non-wandering, U_2 is self positively recursive and there is a $t_3 > 3$ such that $U_2 \cap U_{2_{t_3}} \neq \emptyset$

Choose x_3 in $U_2 \cap U_{2_{t_3}}$ and positive $r_3(\omega) < 1/2^3$ such that $S\,\big(x_3\,, r_3(\omega)\big) \subset U_2 \cap U_{2_{t_3}}$. Set $U\ 3 = S\,(x_3\,, r_3(\omega))$.

We obtain a sequence of open sets $\{U_\lambda\}$, and $U_\lambda \supset U_{\lambda+1}$ and
$\cap\ \{U_\lambda : \lambda = 1, 2, 3, ...\}$ is a singleton {y} with $y \in U$ (prove these facts which are consequences of construction and completeness of X). We claim that y is Poisson stable. To see this notice that by construction, for any integer $\lambda \ge 2, U_\lambda\,(-\,t_\lambda) \subset (\,U_{\lambda-1}) \cdot$ In particular, since $y \in U_\lambda$ for every λ, $y\,(-\,t_\lambda) \in U_{\lambda-1}\cdot$ Clearly, then the sequences $\{y\,(-\,t_{2\lambda})\}$ and $\{y\,(-\,t_{2\lambda+1})\}$ both converge to y. Since $-t_{2\lambda} \to +\infty\ and\ -\,t_{2\lambda+1} \to$-∞
we have $y \in \Gamma^{+}_{y}(\omega)\ and\ y \in \Gamma^{-}_{y}(\omega)$. ■

Theorem 3.20 : Let (ϑ, φ) be an orbitally (positively) Lipschitz stable .If x a point in X is recurrent, then every $x \in \bar{P}$ is non-wandering such that $P \subset X$.

Proof: clearly by theorem (3.17) If x a point in X is recurrent then x is negatively Poisson stable, and by theorem (3.18) then $x \in \bar{P}$ is non-wandering. ■

Corollary 3.21: Let (ϑ, φ) be an orbitally (positively) Lyapunov stable at x.If x a point in X is recurrent, then every $x \in \bar{P}$ is non-wandering such that $P \subset X$.

Proof : by theorem (3. 13 **)** we obtean (ϑ, φ) be an orbitally Lyapunov stable and by theorem(3.20) compleat the proof. ■

4.Conclusion:

I will outline some significant out comes from this research .

i. Interplay Between Stability and Recurrence: Demonstrating that a point is recurrent if and only if it is negatively Poisson stable in an orbitally (positively) Lipschitz stable random dynamical system.

ii. Implications for Non-Wandering Behavior: Showing that a point is intrinsically non-wandering if it is either positively or negatively Poisson stable.

iii. it is shown that for orbitally (positively) Lipschitz stable or Lyapunov stable systems, a recurrent point is non-wandering.
iv. Density of Poisson Stable Points: Providing a mathematical proof that in a complete metric space where every point is non-wandering, the set of Poisson stable points is dense in that space.

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