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Numerical Approach to Solving a Set of Boundary Value Optimal Control Problems Based on Hilbert Space with Reproduction Kernel

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Abstract: The area of failure-free operation of a structure is described by a system of linear inequalities and in the space of loads is a polyhedron. Solving the reliability problem for such areas is very difficult. For the first time in such a formulation, an analytical solution for the reliability function was obtained.

Keywords: structures, load models, linear deformable, analytical solution, construction.

Introduction

This search presents a general method for solving a range of boundary value optimal control problems based on Hilbert theory of reproduction kernel. The status variable as well as the control variable are displayed in series. An iterative method is also used to obtain the corresponding approximate variables, using which we will find the approximate value of the objective function. Under certain conditions the approximate answer will converge uniformly to the analytical answer. The present method is a meshless method and can be easily used. Solving a number of problems with linear and nonlinear dynamic systems have been studied and tested to demonstrate the efficiency and effectiveness of this method.

The contents of this section are taken from the reference [1].

The method presented in this section is a method based on Hilbert space with a reproduction core. The theory of Aronszajn reproduction kernel was first proposed by Zaremba in the early twentieth century.This method is very easy to use and easy to implement.

Hilbert space method with reproduction kernel has been successfully used to solve various problems, of which we can only refer to using them to solve some ordinary differential equations[2,3,4], various types of partial differential equations[5,6,7], different types of integral equations[9,10], some types of inverse problems[8] and even some types of differences equations[2]. In particular, an iterative method based on Hilbert space with a reproduction nucleus is proposed to solve differential equations in Sakar [11]. Cui and Lin [12] can be mentioned as a comprehensive source of Hilbert space-based methods with reproduction kernels to solve.

1- Basic Concepts

In this section, we will introduce some definitions and theorems (without proof)..

Definition 1-1. Suppose Ω is Set Λ ***H*** is Hilbert's real space of fun. $\Upsilon:\Omega \rightarrow IR$, and also $\langle .,. \rangle$is the internal and norm multiplication induced in this space. $||.||=\sqrt{\langle .,. \rangle}$.

The function $K:\Omega\times\Omega \rightarrow IR$ is called the core function of the reproduction *H* if the following conditions are true:

(1) $K_t(.) = K(.,t) \in H, t \in \Omega$

(2) $y(t) = \langle y(.), K_t(.)\rangle, y \in H, t \in \Omega$

Definition 1-2 Hilbert real space H of the functions on set Ω is called a Hilbert space with the reproduction kernel if a function of the reproduction kernel $K$of H exists.

Definition 1-3 Suppose Ω is a Set $\wedge$ K is a IR fun. of the value on $\Omega \times \Omega$. Let K be fun +. if for every $n \in \mathrm{N}$ $\wedge$ every n of the point $\tau_n, \ldots, \tau_1$at Ω, the definite matrix $M = [K(\tau_i, \tau_j)]_{n \times n}$ is negative, meaning that

$$v^T M v = \sum_{i,j=1}^{n} v_i K(\tau_i, \tau_j) v_j \geq 0, \forall v \in \mathrm{R}^n$$

$v \neq 0$, $\rightarrow K$ is definitely definite positive.

Theorem 1-4 Suppose H is a space of the Hilbert function on Ω with the kernel of the reproduction K, $\rightarrow K$ definite fun.+. If for every $n \in \mathrm{N}$ and for every set of distinct points $\tau_n, \ldots, \tau_1$ in Ω, the corresponding set of evaluation points of the functions is linearly independent H;$\rightarrow K$ is definitely definite positive.

Theorem 1-5 (compound trapezoidal integration rule) If $y \in C^2[a, b]$then there is a number like $\eta \in [a, b]$;

$$\int_a^b y(t)\, dt - T(\delta t) = -\frac{(b-a)y''(\eta)}{12}(\delta t)^2$$

where in

$$T(\delta t) = \frac{\delta t}{2}\left(y(\tau^1) + 2\sum_{i=2}^{m-1} y(\tau_i) + y(\tau m)\right)$$

So that

$$\tau_i = a + \frac{(i-1)(b-a)}{(m-1)}$$

For $m \in N$.

2- Implementing the Method

we present a numerical-analytical method based on Hilbert space with a reproduction core to solve a class of optimal boundary value control problems. Objective, to find the optimal control variable $u: \Omega \rightarrow R^1$ and the state variable (vector) $x: \Omega \rightarrow R^d$ that minimize the following objective function:

$$J = h(t_f) + \int_{\Omega} g(t, x(t), u(t))dt, \qquad (1)$$

And apply to a dynamic device as follows:

$$x'(t) = f(t, x(t), u(t)), t \in \Omega, \qquad (2)$$

And also meet the following boundary conditions:

$$x(t_0) = 0, x(t_f) = 0, \qquad (3)$$

Where $d \in N$ and $\Omega = (t_0, t_f)$. We assume that the vector functions f and the scalar functions g and h are, in general, nonlinear and continuous derivatives with respect to their argomans. Sometimes some parts of $x(t_f) \to \infty$ .This method can also be easily used for dynamic devices of higher dimensions [examples 2-3 and 3-3], but for convenience we assume here $d = 1$.

First, we introduce the spaces with the reproduction kernel $W(\Omega)$and $\bar{W}(\Omega)$.

Definition 2-1 suppose

$$W(\Omega) = y{:}\,\Omega \to R \mid y, y' \text{ are absolutely continuous functions on } \Omega \text{ and } y'' \in L^2(\Omega),\ y(t_0) = 0,\ y(t_f) = 0$$

The internal and soft induced multiplication corresponding to the space $W(\Omega)$are defined as follows:

$$(y,z)_w = e_1 y'(t_2)z'(t_0) + e_2 \int_n y'(t)z'(t)dt, y, z \in W(\Omega)$$

$$\| y \|_W = \sqrt{\langle y, y\rangle_W}, y \in W(\Omega).$$

Theorem 2-2 $W(\Omega)$ is a Hilbert space with a reproduction kernel and the function of the reproduction kernel is given as follows:

$$K_s(t) = \begin{cases} l(t,s), & t \le s \\ l(s,t), & t > s \end{cases}$$

Proof: If $\{y_n(t)\}_{n=1}^{\infty}$is a Cauchy sequence in $W(\Omega)$, then

$$\lim_{n\to\infty} \| y_{n+p} - y_n \|_W^2 = \lim_{n\to\infty} \left(e_1(y'_{n+p}(t_0) - y'_n(t_0))^2 + e_2 \int_\Omega (y''_{n+p}(t) - y''_n(t))^2 dt \right) = 0$$

So we have

$$\lim_{n\to\infty} | y'_{n+p}(t_0) - y'_n(t_0) | = 0,$$

$$\lim_{n\to\infty} \int_\Omega (y''_{n+p}(t) - y''_n(t))^2 dt = 0.$$

Which shows that the sequences $\{y'_n(t_0)\}_{n=1}^{\infty}$and $\{y''_n(t)\}_{n=1}^{\infty}$ are the Cauchy sequences in R and $L^2(\Omega)$, respectively. Then E μ unique real& function unique $Z \in L^2(\Omega)$;

$$\lim_{n\to\infty} y'_n(t_0) = \mu, \lim_{n\to\infty} \int_\Omega (y''_n(t) - Z(t))^2 dt = 0.$$

If

$$y(t) = \mu(t - t_0) + \int_{t_0}^{t} \int_{t_0}^{t_1} Z(s)\, ds\, dt_1,$$

Then

$$y'(t) = \mu + \int_{t_0}^{t} Z(s)\, ds, y''(t).$$

So $y \in W(\Omega)$ on the other hand, we have $y'(t_0) = \mu$ so

$$\lim_{n\to\infty} \| y_n - y \|_W^2 = e_1 \lim_{n\to\infty} (y_n'(t_0) - y'(t_0))^2 + e_2 \lim_{n\to\infty} \int_\Omega (y_n''(t) - y''(t))^2 dt$$
$$= e_1 \lim_{n\to\infty} (y_n'(t_0) - \mu)^2 + e_2 \lim_{n\to\infty} \int_\Omega (y_n''(t) - Z(t))^2 dt = 0.$$

The result $W(\Omega)$ a Hilbert space. Now it suffices to prove that the fun. $K_s(t) \in W(\Omega)$ exists such that

$$\langle y(t), K_s(t)\rangle_W = y(s), \forall s \in \Omega \qquad \forall y \in W(\Omega).$$

Suppose we have $K_s(t) \in W(\Omega)$then

$$\langle y(t), K_s(t)\rangle_W = e_1 y'(t_0)\, \partial_t K_s(t_0) + e_2 \int_\Omega y''(t)\, \partial_t^2 K_s(t)\, dt$$
$$= e_1 y'(t_0)\, \partial_t K_s(t_0) + e_2 y'(t)\, \partial_t^2 K_s(t)]_{t_0}^{t_f} - e_2 \int_\Omega y'(t)\, \partial_t^3 K_s(t)\, dt$$
$$= e_1 y'(t_0)\, \partial_t K_s(t_0) + e_2 y'(t)\, \partial_t^2 K_s(t)]_{t_0}^{t_f} - e_2 y(t)\, \partial_t^3 K_s(t)\, |_{t_0}^{t_f}$$
$$+ e_2 \int_\Omega y(t)\, \partial_t^4 K_s(t)\, dt$$
$$= y'(t_0)\big(e_1\, \partial_t K_s(t_0) - e_2\, \partial_t^2 K_s(t_0)\big) + e_2 \sum_{i=0}^{1} (-1)^{1-i} y^{(i)}(t_f)\, \partial_t^{3-i} K_s(t_f)$$
$$- e_2 \int_\Omega y(t)\, \partial_t^4 K_s(t)\, dt$$

Therefore, $K_s(t)$ answer to the following generalized differential equation:

$$\begin{cases} \partial_t^4 K_s(t) = -\dfrac{1}{e_2}\delta(t - s), \\ e_1\, \partial_t K_s(t_0) - e_2\, \partial_t^2 K_s(t_0) = 0, \\ \partial_t^2 K_s(t_f) = 0. \end{cases}$$

then

$$\begin{cases} \partial_t^4 K_s(t) = 0, t \neq s \\ e_1\, \partial_t K_s(t_0) - e_2\, \partial_t^2 K_s(t_0) = 0 \\ \partial_t^2 K_s(t_f) = 0 \\ K_s(t_0) = 0 \\ K_s(t_f) = 0 \\ \lim_{t \to s^0} \partial_t^i K_s(t) = \lim_{t \to s^-} \partial_t^i K_s(t), i = 0,1,2 \\ \lim_{t \to s^0} \partial_t^3 K_s(t) = \lim_{t \to s^-} \partial_t^3 K_s(t) = -\frac{1}{e^2} \end{cases}$$

It can be shown that the answer to this equation is as follows:

$$\begin{cases} l(t,s), t \leq s, \\ l(s,t), t > s. \end{cases}$$

Where l is a function referred to in the theorem.

Obviously, the point evaluations of the functions on $W(\Omega)$are linearly independent because the space of polynomials is a subset of the space $W(\Omega)$.

Definition 2-3 suppose

$$\bar{W}(\Omega) = y : \Omega \to R \mid y' \in L^2(\Omega) \text{ and } Y \text{an absolutely cont. fun.}$$

The induced internal and norm multiplications for $\overline{W}(\Omega)$are defined as follows:

$$\langle y, z \rangle_{\overline{W}} = b_1 y(t_0) z(t_0) + b_2 \int_\Omega y'(t) z'(t)\, dt, y, z \in \overline{W}(\Omega)$$
$$\| y \|_{\overline{W}} = \sqrt{\langle y, y \rangle_{\overline{W}}}, y \in \overline{W}(\Omega) \qquad (4)$$

Theorem 2-4 $\overline{W}(\Omega)$is a Hilbert space with a reproduction kernel and the function of its reproduction kernel is as follows:

$$\bar{K}_s(t) = \begin{cases} \bar{l}(t,s), t \leq s, \\ \bar{l}(s,t), t > s. \end{cases}$$

where in

$$\bar{l}(t,s) = \frac{-b_1 t_0 + b_2 + t b_1}{b_1 b_2}$$

Proof: Similar to the theorem 2-2.

Theorem 2-5 If we assume that the analytical answers (2-2) and (2-3) mean that x belongs to $W(\Omega)$and $\{\tau_i\}_{i=1}^{\infty}$is condensed in Ω, then

$$x(t) = \sum_{j=1}^{\infty} c_j\, \psi_j(t) \qquad (6)$$

The coefficients c_j are calculated by solving the system of semi-infinite equations:

$$\boldsymbol{Ac} = \boldsymbol{e} \tag{7}$$

where in

$$\boldsymbol{A} = [(\psi_i(.), \psi_j(.))_W]_{i,j=1,2,\dots}, c = [c_1\ c_2\ \dots]^T,$$
$$e = [f(\tau_1, x(\tau_1), u(\tau_1))\ \ f(\tau_2, x(\tau_2), u(\tau_2))\ \ \dots]^T.$$

Proof: If for any integer $n$we have

$$\sum_{j=1}^{n} a_j \psi_j(t) = 0, t \in \Omega,$$

Then from (2-4) it results that

$$\sum_{j=1}^{n} a_j L_s K_s(t) |_{s=\tau_j} = 0, t \in \Omega,$$

since L_s is (linear operator), we have:

$$L_S \left(\sum_{j=1}^{n} a_j K_s(t) \right) |_{s=\tau_j} = 0, \forall t \in \Omega.$$

L_s is inverse and then we have:

$$\sum_{j=1}^{n} a_j K_s(t) |_{s=\tau_j} = 0, \forall t \in \Omega.$$

By choosing $t = \tau_i$ for $i = 1, \dots, n$ we can deduce that $\alpha^T M \alpha = 0$, where $\alpha = [a_1, \dots, a_n]^T$and $M = [K(\tau_i, \tau_j)]_{n \times n}$. Since the nucleus of reproduction K is strictly positive, we conclude that $\alpha = 0$. As a result, $\dots, \psi_2, \psi_1$ are linearly independent, and based on Theorem (2-2), $\dots, \psi_2, \psi_1$ is a basis for the space $W(\Omega)$. So $x \in W(\Omega)$has a representation in the form (2-5). By placing (2-5) in (2-2) we have:

$$\sum_{j=1}^{\infty} c_j L_t \psi_j(t) |_{t=\tau_i} = f\big(\tau_i, x(\tau_i), u(\tau_i)\big), i = 1,2, \dots$$

Because

$$L_t \psi_j(t) |_{t=\tau_i} = (\phi_i(.), L_t \psi_j(.))_{\bar{W}} = (L_t^0 \phi_i(.), \psi_j(.))_W = (\psi_i(.), \psi_j(.))_W.$$

And the proof is complete.

If d is linear in equation (2-2) with relative x. In other words, the device (2-6) will be a linear device and the coefficients $\{c_j\}_{j=1}^{m}$will be calculated by solving this device. Otherwise we have to create an iterative method to solve the device (2-6) to find the approximate status. For this purpose, we consider the order of repetition as m and set the initial state variable to $x_0 = 0$ based on the initial conditions. The approximate result of the state variable x is represented as follows:

$$x_m(t) = \sum_{j=1}^{m} c_j\,\psi_j(t), \tag{8}$$

Where the coefficients $\{c_j\}_{j=1}^{m}$ are obtained from solving the following device:

$$\sum_{j=1}^{m} c_j\, L_t\psi_j(t)\,|_{t=\tau_i} = f\big(\tau_i, x_{m-1}(\tau_i), u(\tau_i)\big), i = 1, \dots, m, \tag{9}$$

Now we have to remember that x_m (approximate state variable) depends on the unknown values of $u(\tau_m), \dots, u(\tau_1)$.

$$then \quad J_m\big(u(\tau_1), \dots, u(\tau_m)\big) = h(t_f) + T(\delta t), \tag{10}$$

Where $T(\delta t)$is similar to what is presented in Theorem 1-3 with the following conditions:

$$a = t_0, b = t_f, y(t) = g\big(t, x_m(t), u(t)\big). \tag{11}$$

J_m is an approximation of the value of the objective function J, and by minimizing it, the approximate values of $u(\tau_m), \dots, u(\tau_1)$can be obtained. Then, using Equations (9) and (11), we obtain the approximate answer of x_m and the approximate value of the objective function. This process is summarized in the following algorithm:

1. For $m = 1,2, \dots$, we select $m$point on Ω.
2. We put $\psi_j(t) = L_s K_s(t)\,|_{s=\tau_j}, j = 1, \dots, m.$
3. We put $A = [(\psi_i(.), \psi_j(.))_W]_{i,j=1,2,\dots,m}$ Or $A = [L_t\psi_j(t)\,|_{t=\tau_i}]_{i,j=1,2,\dots,m}$.
4. We obtain the LUdecomposition of the matrix of coefficients A.
5. Set the initial value to $x_0 = 0$ and $j = 1$.
6. We place $e_j = \left[f\left(\tau_1, x_{j-1}(\tau_1), u(\tau_1)\right) \dots f\left(\tau_j, x_{j-1}(\tau_j), u(\tau_j)\right)\right]^T$ and subtract the matrix A_j from the principal components j of the row and the first column of the Matrix A.
7. **Solve** the equation $\boldsymbol{A_j c = e_j}$ using the LUdecomposition of the matrix A.
8. We put $x_j(t) = \sum_{i=1}^{j} c_i\,\psi_i(t)$.
9. We put $j = j + 1$. If $j \le m$we return to step (6).
10. So, $u(\tau_m), \dots, u(\tau_1)$and $x_m$are minimized by placing them in Equation (11).
11. We get the approximate answers x_m and J_m.

Theorem 2-6 Suppose $\{\tau_i\}_{i=1}^{\infty}$ **is dense in** Ω. Then the approximate answer x_m will converge to the analytical answer x.

Proof: Since $\| \frac{dK_s(t)}{dt} \|_W$ is bounded [8], then according to Arzela–Ascoli theorem $P = \{x \mid \ \| x \|_W \leq \gamma\}$ for a positive constant value γ will be a compact set at space $C(\Omega)$. Thus $\{x_m\}_{m=1}^{\infty}$ will be a sequence of finite functions in $W(\Omega)$. Consequently for any i, there exists a convergent subset of $\{x_m(\tau_i)\}_{i=1}^{\infty}$ from $\{x_m(\tau_i)\}_{m=1}^{\infty}$ such that uniformly $x_m(\tau_i) \to x(\tau_i) when\ i \to \infty$.

Then

$$L_t x_m(\tau_i) = f\big(\tau_i, x_{m-1}(\tau_i), u(\tau_i)\big).$$

So we have:

$$L_t x_m(\tau_i) = f\big(\tau_i, x_{m-1}(\tau_i), u(\tau_i)\big), \quad (11)$$

By taking the limit from both sides of the equation (2-12), we can write:

$$L_t x(\tau_i) = f\big(\tau_i, x(\tau_i), u(\tau_i)\big).$$

Since $\{\tau_i\}_{i=1}^{\infty}$is dense in Ω, we have:

$$L_t x(t) = f\big(t, x(t), u(t)\big).$$

For all $t \in \Omega$, in other words, x is definitely the analytical answer (2-2).

Theorem 2-7 If $g\big(., x(.), u(.)\big)$is continuously derivative of two terms, then J_m, which is obtained from the present method, when $m \to \infty$tends to J.

Proof: Based on equation (2-11) and theorems (1-5) and (2-6) when we have

$m \to \infty$ or $\delta t \to 0$:

$$x_m \to x, T(\delta t) \to \int_{t_0}^{t_f} g\big(t, x(t), u(t)\big) dt.$$

So $J_m \to J when m \to \infty$.

3- Numerical Examples

we use the present method to numerically solve a category of optimal limit value control problems.. We use the infinitely discrete norm and the norm average sum of squares, i.e

$$E_\infty = \max_{1 \leq i \leq m} \mid w(\tau_i) - \bar{w}_i \mid, E_2 = \left(\sum_{i=1}^{m} \frac{(w(\tau_i) - \bar{w}_i)^2}{m} \right)^{\frac{1}{2}}$$

Where(w)and($\bar{w}$), exact and approximate solutions.

Example 3-1 We want to find the optimal control $u$that minimizes the following problem:

$$J=\int_0^1 (u^2(t)+x^2(t))dt,$$

s.t.

$$x'(t)=u(t) \qquad ,x(0)=0 \qquad ,x(1)=0/5 \rightarrow J=0/3282588214,$$

and the corresponding exact answers are as follows [15]:

$$x^*(t)=\frac{\sin \mathrm{h}(t)}{2\sin \mathrm{h}(1)}, u^*(t)=\frac{\cos \mathrm{h}(t)}{2\sin \mathrm{h}(1)}.$$

Example 3-2 Our goal is to find the optimal control so that the following objective function is minimized:

$$J=\frac{1}{2}\int_{t_0}^{t_f} u^2(t)\, dt,$$

According to the following dynamic device:

$$\begin{cases} x_1'(t)=x_2(t), \\ x_2'(t)=C\sin(kt)+u(t), \end{cases}$$

And according to the following border conditions:

$$x_1(0)=0, x_2(0)=0, x_1(t_f)=B, x_2(t_f)=0.$$

The exact answer can be obtained using the Pontriagin maximum principle as follows [7]:

$$x_1^*(t)=-\left(\frac{C}{k^2}\right)\sin(kt)+c_1\left(\frac{t^3}{6}\right)+c_2\left(\frac{t^2}{2}\right)+\left(\frac{C}{k}\right)t,$$

$$x_2^*(t)=-\left(\frac{C}{k}\right)\cos(kt)+c_1\left(\frac{t^2}{2}\right)+c_2 t+\left(\frac{C}{k}\right),$$

$$u^*(t)=-c_1 t-c_2,$$

where in

$$c_1=-\left(\frac{6}{t_f^2}\right)\left(\left(\frac{C}{k}\right)\cos(kt_f)-\left(\frac{C}{k}\right)+\left(\frac{12}{t_f^3}\right)B+\left(\frac{C}{k^2}\right)\sin(kt_f)-\left(\frac{C}{k}\right)t_f\right)$$

$$c_2=-\left(\frac{2}{t_f}\right)\left(\left(\frac{C}{k}\right)\cos(kt_f)-\left(\frac{C}{k}\right)+\left(\frac{6}{t_f^2}\right)B+\left(\frac{C}{k^2}\right)\sin(kt_f)-\left(\frac{C}{k}\right)t_f\right)$$

We also assume , $C=0.1 \quad ,k=8 \quad ,t_f=10 \quad ,B=1 \rightarrow J=0.005343386825.$

Example 3-3

The purpose of this problem is to find the optimal control u that minimizes the following function:

$$J=x_2(t_f),$$

According to the following nonlinear dynamic device:

$$\begin{cases} x_1'(t) = 0/5x_1(t) + u(t), \\ x_2'(t) = x_1^2(t) + 0/5u^2(t). \end{cases}$$

With border conditions $x_1(t_0) = 1, x_2(t_0) = 0, x_1(t_f) = 0/5$. The exact answers are as follows [13]:

$$x_1^*(t) = a_1 e^{t/5} + a_2 e^{-t/5},$$
$$x_2^*(t) = a_3 e^{3t} - a_4 e^{-3t} + c_1,$$
$$u^*(t) = -a_5 e^{t/5} - a_6 e^{-t/5},$$

where in

$$a_1 = \frac{1/2 - e^{-7/5}}{e^{7/5} - e^{-7/5}}, a_2 = 1 - a_1, a_3 = 1/2a_1^2, a_4 = a_2^2,$$
$$a_5 = -a_1, a_6 = 2a_2, c_1 = -a_3 + a_4$$
$$; t_0 = 0 \wedge t_f = 5 \text{ then } J = \frac{1}{124170946758}.$$

Example 3-4

Find the optimal control u that minimizes the following problem:

$$J = \int_0^5 (x^2(t) + 0.5u^2(t))\, dt,$$

With the following conditions:

$$x'(t) = 0/5x(t) + u(t),$$
$$x(0) = 1, x(5) = 0/5.$$

In Table (3-2), the absolute error corresponding to the value of the objective function J, $E_2 \wedge E_\infty$ to $u \wedge x$ and a number of different points are listed. Numerical results confirm the results of the theory.

Table 3-2 Absolute error J, $u \wedge x$ to Ex.(3-3)

m	$\lvert J - Jm \rvert$	$E^2(u)$	$E\infty(u)$	$E^2(x)$	$E\infty(x)$
16	$4/1E-5$	$2/3E-2$	$6/8E-2$	$8/4E-5$	$8/8E-4$
32	$2/3E-6$	$5/5E-3$	$1/9E-2$	$5/1E-6$	$1/1E-4$
64	$1/6E-7$	$1/4E-3$	$4/9E-3$	$5/4E-7$	$1/3E-5$
128	$1/7E-8$	$3/4E-4$	$1/2E-3$	$1/5E-7$	$1/6E-6$

Example 3-5 The purpose of this example is to find the optimal control $u$so that it minimizes the following function:

$$J = \int_0^1 ((2 - x(t))^2 + u^2(t))dt,$$

According to the following nonlinear dynamic device:

$$x'(t) = -0/25\sqrt{x(t)} + u(t).$$

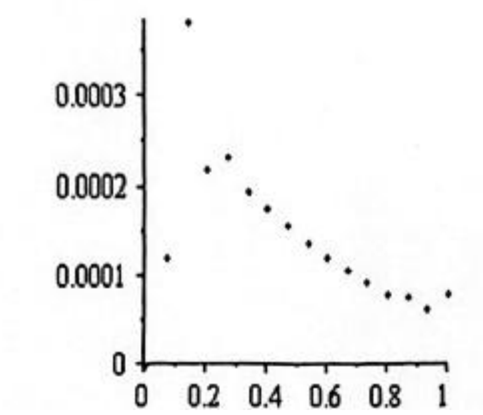
And the following border conditions:

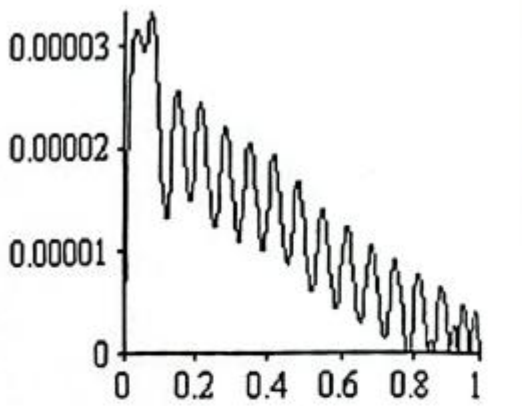
$$x(0) = 0, x(1) = 2.$$

The exact answers are as follows: [14]

$$\begin{cases} x^*(t) = \dfrac{e^{t+1} + 63e^t - 63e^{-t+2} - e^{-t+1} + 63e^2 - 63}{32(e^2 - 1)}, \\ u^*(t) = x'^*(t) + 0/25\sqrt{x^*(t)}. \end{cases}$$

The absolute error for $u$and $x$is shown in the diagram (1-1). Table (3-2) lists the errors related to approximate answers and results.





(u) **Figure 1-1** (x)

Table 3-2. Comparison of the present method with other methods to Ex.(3-5)

Method	*Max. error(u)*	*Max. error(x)*
Neural network method [9]	0.1	0/001
Fuzzy method	0/03 4E − 5	−
The present method	0/0004	3/5E − 5

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