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A New Thermodynamic Limit for the [(3·N-1)/2, N/2)] Collatz Sequence

Danilo Merlini [1, 2, 3], Luca Rusconi [1,3], Massimo Sala [2, 3], Nicoletta Sala [2, 3]
[1] CERFIM (Research Center for Mathematics and Physics), Locarno, Switzerland
[2] IRFIA (Research Institute in Arithmetic Physics), Locarno, Switzerland
[3] ISSI (Institute for Scientific and Interdisciplinary Studies), Locarno, Switzerland

Abstract: We consider a new way to analyse the cardinality of the odd and of the even numbers emerging in the above Collatz sequence and we find that as the chalice of the inverse orbits grows to infinity the cardinality of them are the same, i.e. we have that (in particular):

$$\lim_{N\to\infty} \frac{n_{odd}}{n_{ev}} = 1$$

a limit which is different from the usual one with a vertical depth.

Keywords: Cycle, Collatz Sequence, Even and Odd Numbers, Thermodynamic Limit.

1. Introduction

This sequence has three cycles (See [1], [2], [3]) and as a function k (the vertical depth) of the chalice of the inverse orbits ,the ratio between the total number of the odd numbers up to k and that of the even numbers is given as a function of the growing constant c=4/3 by c-1 =1/3 for some sequences [2, 3]. In a recent work concerning a (formal) analogy with the parity invariance in Physics (Lee and Yang [4]), it was analysed [3] as our suggestion of [4] the solutions of odd→odd for the above sequence in solving the equations:

$$\frac{(3 \cdot d_1 - 1)}{2^k} = d_2 \qquad (1)$$

with d_1 and d_2 any two odd numbers smaller or at most equal to the integer N, large at our convenience. The analytical results are given below - here up to k=21 - with the emergence of two sets of integers for d_2, the first at k=1 given by 1+6·p, p=0...p_m where p_m(k=1) (the maximum integer) is given by the floor [(N-1)/4] and where the first set for d_1 is given by 1+4·p, p=0.. p_m(k=1) . Table 1 contains the solutions of Eq.(1) up to k=21.

k	d_1	d_2
1	$1+4\cdot p = 1+2^2\cdot p$	$1+6\cdot p$
2	$7+8\cdot p = 7+2^3\cdot p$	$5+6\cdot p$
3	$3+16\cdot p = 3+2^4\cdot p$	$1+6\cdot p$
4	$27+32\cdot p = 27+2^5\cdot p$	$5+6\cdot p$
5	$11+64\cdot p = 11+2^6\cdot p$	$1+6\cdot p$
6	$107+128\cdot p = 107+2^7\cdot p$	$5+6\cdot p$
7	$43+256\cdot p = 43+2^8.\ p$	$1+6\cdot p$
8	$427+512\cdot p = 427+2^9\cdot p$	$5+6\cdot p$
9	$171+1024\cdot p = 171+2^{10}\cdot p$	$1+6\cdot p$

10	$1707+2048\cdot p = 1707+2^{11}\cdot p$	$5+6\cdot p$
11	$683+4096\cdot p = 683+2^{12}\cdot p$	$1+6\cdot p$
12	$6827+8192\cdot p = 6827+2^{13}\cdot p$	$5+6\cdot p$
13	$2731+16384\cdot p = 2731+2^{14}\cdot p$	$1+6\cdot p$
14	$27307+32768.p = 27307+ 2^{15}.p$	$5+6\cdot p$
15	$10923+65536\cdot p = 10923+ 2^{16}\cdot p$	$1+6\cdot p$
16	$109227+131072 = 109227+2^{17}\cdot p$	$5+6\cdot p$
17	$43691 +262144 = 43691 + 2^{18}\cdot p$	$1+5\cdot p$
18	$436907 +524288 = 436907+ 2^{19}\cdot p$	$5+6\cdot p$
19	$174763+ 1048576= 174763 + 2^{20}\cdot p$	$1+6\cdot p$
20	$1747627+2097152=1747627+ 2^{21}\cdot p$	$5+6\cdot p$
21	$699051+4194304= 699051 + 2^{22}\cdot p$	$1+6\cdot p$

Table 1. The solutions of Eq.(1) up to k=21.

In general $a_k + b_k\cdot p$, $p=0..p_m(k)$, and $p_m(k) = \text{floor}((N-a_k)/b_k)$ (In the Sequel we omit the index of the cascades of the even numbers (notice that for $k=1, a_1=1$ and $b_1=4$).

2. Numerical experiments

Example 1

As a first example we consider N=136 (the maximum integer in the 3 cycles (it belongs to C_3).

We subsum ,in the Table 2, all the odd and all the even numbers at most equals to N=136.

In the third column we add the numbers from a model (geometric progression of ratio q=1/2) starting at $[d_1(k=1)]=N/4$, explained in Section 3.

k	$d_1 = d_{od}$	d_{ev}	d_{th}
1	34	36	34=136/4
2	17	15	17
3	9	10	8.5
4	4	3	4.25
5	2	3	2.125
6	1	0	1.0625
7	1	1	0.503125
8			0.2515
9			0.125
total	d_{od} = 68=N/2	d_{ev} = 68=N/2	d_{th} = 67.3...

Table 2. $d_1(k) = d_{od}(k)$, $d_{ev}(k)$, $d_{th}(k) = \frac{N}{2^{k+1}}$ for N=136. Remark: if we increase N by 1, i.e. N=137, then $d_1(k=1)=15$, thus $d_{od}(k)$, $d_{od}=69$, $d_{ev}=68$ and 69+68=137 (notice that the orbit of 137 end up on C_3).

Example 2

As a second example we consider N=4096. The number d_{od} =sum(over k) of $d_1(k)$ and of the even numbers are also obtained with the formula (2) (the set of all even numbers above any d_1 i.e. $[d_1] = [(a+b\cdot p)]$, d_{ev} are given below in table 3.

$$d_{od} = \sum_k [d_1] = \sum_k \sum_{p=0}^{p_m} \left[floor\left(\frac{N-a}{b}\right) + 1 \right]$$

$$d_{ev} = \sum_{k}\sum_{p=0}^{pm}\left[floor\left(\frac{1}{log2}\right)\cdot\left[log\left(\frac{N}{a+b\cdot p}\right)\right]\right] \quad (2)$$

k	$d_{od}(k)$	$d_{ev}(k)$
1	1024	1025
2	512	511
3	256	257
4	128	127
5	64	65
6	32	31
7	16	17
8	8	7
9	4	5
10	2	1
11	1	2
12	0	0
13	1	0
total	d_{od} = 2048 =4096/2	2048=4096/2

Table 3. $d_1(k) = d_{od}(k)$ and $d_{ev}(k)$ for N=4096.

Example 3

Let now N=10000.

k	$d_1(k) = d_{od}(k)$	$d_{ev}(k)$	$\Delta = d_{ev}-d_{od}$	d_{th} (from the model)
1	2500	2503	+3	2500
2	1250	1246	-1	1250
3	625	629	+3	625
4	312	309	0	312.5
5	157	159	+2	156.25
6	78	75	-1	78.125
7	39	42	+2	39.0625
8	19	17	0	19.5312
9	10	12	+2	9.7656
10	5	3	0	4.8828
11	3	4	+1	2.4414
12	1	0	0	0.6103
13	1	1	0	
total	d_{od} = 5000 ==10000/2=N/2	d_{ev} = 5000 = = 10000/2=N/2		d_{th} = 4999.3896...

Table 4. $d_{od}(k)$ and $d_{ev}(k)$, $\Delta(k) = \sum_{i=1}^{k} d_{ev}(i) - d_{od}(i)$ and $d_{th}(k) = \frac{N}{2^{k+1}}$ for N=10000.

Above we have given the fluctuations Δ (up to k) between the total [d_{ev}]-[d_{od}] i.e. up to k. The model with q=1/2, where $d_1(k)=N/(2^{k+1})$, N/4 for k=1, gives 4999.3896.. up to the last term 0.6103..

For the last example we limit us to give the amount of [d_1] (the odd numbers) up to k=23 of the number $N=10^6$ and the total amount.

k	$d_1(k)= d_{od}(k)$
1	$N/2^2$ =N/4 = 250000
2	$N/2^3$ =N/8 = 125000

3	$N/2^4$ =N/16 = 62500
4	$N/2^5$ =N/32 = 31250
5	$N/2^6$ =N/64 =15625
6	$N/2^7$ =N/128=7812.50
7	$N/2^8$ = N/256= 3906.25
8	$N/2^9$ = N/512= 1953.25
9	$N/2^{10}$= N/1024= 976.5625
10	$N/2^{11}$= N/4096= 488.28125
11	$N/2^{12}$ = 244.140625
12	$N/2^{13}$ =122.073125
13	$N/2^{14}$ = 61.03515625
14	$N/2^{15}$ = 30.51757812
15	$N/2^{16}$ = 15.258789
16	$N/2^{17}$ = 7.629394
17	$N/2^{18}$ = 3.814697
18	$N/2^{19}$ = 1.907348
19	$N/2^{20}$ = 0.953674
20	$N/2^{21}$ = 0.476837
21	$N/2^{22}$ = 0.238418
total 500000 total 4999999,7615.. ~ 500000 = N/2	

Table 5. For k=21, the last term is 0.2384...= 500000-499999.7615.

I.e. $\sum_{k=1}^{21} = \frac{N}{2}\left(1-\left(\frac{1}{2}\right)^{21}\right) = \frac{10^6}{2}(1-0.2384) = 49999.7615.$

3. Inverse orbits

As recalled above, it is known that there are 3 cycles (at least, but the conjecture is that their number is at most 3 [3, 4, 5, 6].

They are:

C_1: (1→1)

C_2: (5 → 7→ 10→5)

C_3: (17→25→37→55→82→41→61→91→136→68→34→17)

of length 1, 3, resp. 11.

The sets of numbers which fall into the three cycles are of course disjoint

but all the three cycles contribute to the emergence of the total set of the odd and of the even numbers in Equations (1). (See next Section).

We notice in fact that for C_1, we have (d_1, d_2) = (1→1) (k=1) and in the set 1+6·p, p=0, 1+4·p=d_1=1.

for C_2: (d_1→d_2)= (5→7) (k=1) and in the set 1+6·p, p=1, 1+4·p=d_1= 5, (d_1→d_2) = (7→5), (k=2), and in the set 5+6·p, p=0 , 7+8·p=d_1= 7.

For the last cycle C_3 we add the table 6.

17→25 , (k=1) p=4 in d_2= 1+6·p
25→37, (k=1) p=6 in d_2= 1+6·p
37→55, (k=1) p=9 in d_2= 1+6·p
55→82→41 (k=2),p=6 in d_2= 5+6·p
41→61 (k=1), p=10 in d_2= 1+6·p

$61 \to 91$ (k=1), p=15 in $d_2=1+6\cdot p$
$91 \to 136 \to 68 \to 34 \to 17$, (k=4), p=2 in $d_2=5+6\cdot p$ ($d_1=27+32\cdot p= 27+2\cdot 32 = 91$)

Table 6. The last cycle C_3

Thus, the three cycles contribute by means of the two sets of $d_2 = 1+6\cdot p$ (k odd) or $5+6\cdot p$ (k even) for some p. In Fig. 1 we illustrate the sequence up to n=136 with the three cycles on Figures.

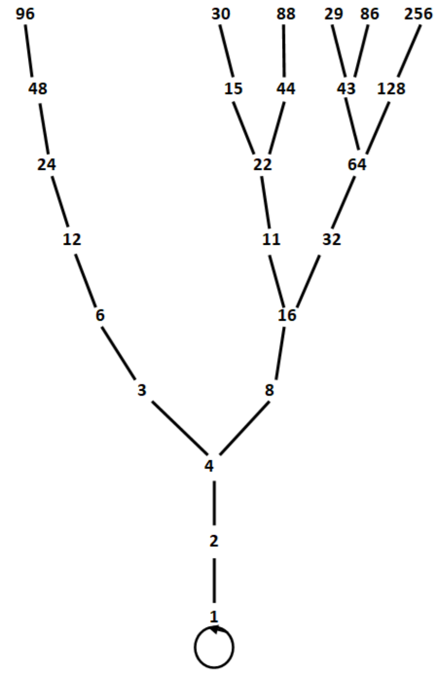

Fig.1 C_1= (1, 1).

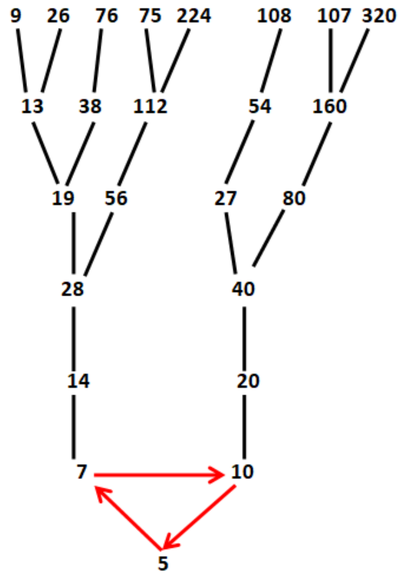

Fig. 2 C_2 = (5, 7, 10, 5).

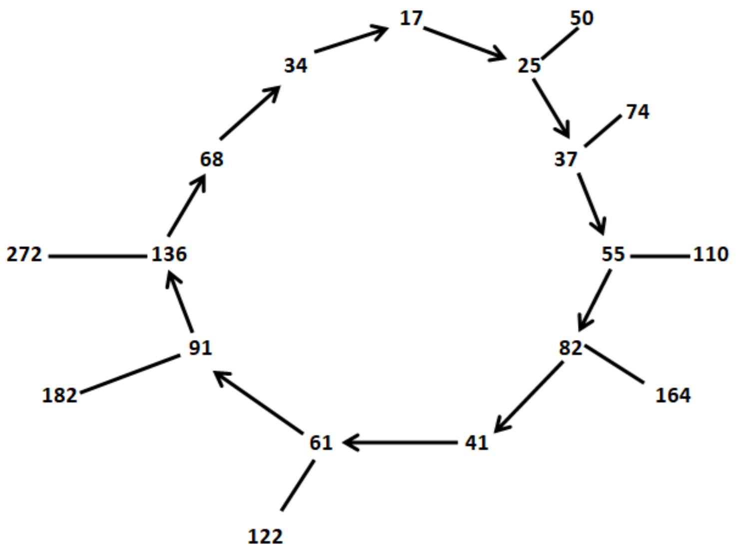

Fig. 3 C_3= (17, 25, 37, 55, 82,41, 61, 91 136, 68, 34, 17).

4. The model

The model [7] which also holds here, is suggested by the results of the numerical experiments given above in the Tables i.e. for a chalice of the inverses orbits, with N large at our convenience, where the first term a_1=N/4 is the sum of the other infinite terms (1024 for N=4096 and 2500 for N=10000, 250000 for N=10^6). Notice that the cascades of the even (factor $1/2^k$, i.e. k, is now allowed to extends up to infinity, i.e. N→∞).

For the cardinality of the set [d_1] we solve $a+b\cdot p_m = N$, where p_m is the maximum integer and the sum is carried out with the floor function; the same for the even numbers where the equation is

$(a+b\cdot p)\cdot 2^{n(p)} \leq N$, with n(p) an integer. For an upper bound (for the even at k=1), we use some controlled approximations and also for a "lower bound" i.e. floor(x) =x*+ε(x)*, for some ε(x*).

Then, application of the Stirling formula.

Finally, assuming a geometric progression with ratio q=1/2 (as in Ref. [7]), our finding is that (with q=1/2):

$$\sum\left(\frac{N}{4}+\frac{N}{8}+\cdots\right)=\sum\left(\frac{N}{2^{k+1}}, k=1\ldots\infty\right)=\left(\frac{N}{4}\right)\cdot\left(\frac{1}{1-\frac{1}{2}}\right)=\frac{N}{2} \qquad (3)$$

(in particular) as N→∞ the totalities of the odd and of the even numbers is the same.

The result apply to other Collatz Sequences: an interesting case is given by [(3·n+7)/2, n/2)] where

extended numerical experiments (in particular) indicate only one cycle i.e. (5→11→20→10→5). [8] . Of course there is a second cycle disjoint of the above one i.e. the multiple of 7 with orbits ending in the cycle 7 (1→2→1) = (7→14→7) [6]. The conjecture is that only the above two cycles exist, the first being the arrival of the orbits of all integers not multiples of 7, and the second being the arrival of the orbits of all integers of 7.

5. Concluding remark

For clarity, it should be noticed that our results do not contradict the usual limit obtained by means of a vertical depth with a constant (for the (3n+1)/2), c-1, i.e., 1/3), different from 1. Our finding indicates that in some Collatz sequences a limit is different in the way one analyzes a "depth".

A simple example may illustrate this point.

If in the sequence ((3n+1)/2, n/2) (not (3n-1)/2) [7]) we consider N=27, then the length l of the orbit of N=27 i.e. l(27)=71 (vertical depth =71), while in the above formulation with Eq.(1), N=27 appears already at k=1 for the usual sequence i.e. [(3·n+1)/2, n/2] since we have $((3 \cdot d_1+1)/2)) =$ $=(3 \cdot 27+1)/2 = 41$.

For many decades, scholars have sought to identify interesting connections between the Collatz sequence and other problems [6, 7, 8, 9, 10, 11,12].

In this work, we have proposed an alternative method for handling odd and even numbers, specifically by using even-number cascades. Our result leaves open the problem of whether, in the [(3•n+1)/2, n/2] other cycles exist in addition to the cycle (1, 2, 1) [7], but supports the absence of trajectories (orbits) going to infinity in another way than that in our limit.

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