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Formulas for a Cyclic Quadrilateral

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Abstract: This article discusses one of the well-known theorems of elementary geometry Ptolemy's theorem along with several of its proofs. It also provides suggestions and guidelines for using these proofs to establish other related theorems.

Keywords: Theorem, Ptolemy, Carnot, Pythagoras, Main, Bretschneider, Case.

Introduction. It is well known from the secondary-school mathematics curriculum that the study of mathematics involves working with statements known as theorems. Properties of concepts that are not fundamental and are not included in their definitions are usually established by proof. Such provable properties of concepts are called theorems. The word theorem originates from Greek, and its literal meaning is "to observe" or "to contemplate." Therefore, in the school mathematics curriculum, a theorem is defined as follows [1].

Definition. A mathematical statement that requires proof is called a *theorem*. In what follows, we consider several proofs of one such statement Ptolemy's theorem.

Review of the Relevant Literature. Both in our country and abroad, significant contributions have been made to the proofs of Ptolemy's theorem and to solving complex problems derived from it by researchers such as T.R. Tulegenov, S.Kh. Abjalilov, A. Arziqulov, Dasari Naga, Vijay Krishna, Mihai Miculi, and others. In the works of the aforementioned scholars, various consequences of the theorem and certain approaches to solving complex problems arising from it have been proposed. Taking these approaches into consideration, we present several different methods for proving this theorem, intended for those interested in solving olympiad-level problems.

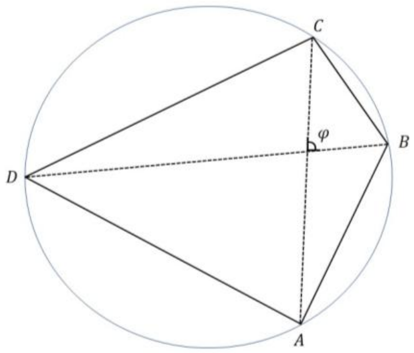
Research Methodology.

Theorem (Ptolemy). In a quadrilateral inscribed in a circle, the product of its diagonals is equal to the sum of the products of its two pairs of opposite sides.

Proof.

Method 1 (Using the Law of Cosines, Figure 1).

Figure 1.



$$\Delta ABC: AB^2 = AE^2 + BE^2 + 2AE \cdot BE \cdot cos\varphi,$$

$$\Delta CEB: BC^2 = CE^2 + BE^2 - 2CE \cdot BE \cdot cos\varphi,$$
$$\Delta CED: DC^2 = DE^2 + CE^2 + 2DE \cdot CE \cdot cos\varphi,$$
$$\Delta AED: AD^2 = DE^2 + AE^2 - 2DE \cdot AE \cdot cos\varphi.$$

From the given equations, we obtain the following:

$$AB^2 - BC^2 + DC^2 - AD^2 = 2\big(BE(AE + CE) + DE(CE + AE)\big)cos\varphi,$$
$$a^2 - b^2 + c^2 - d^2 = 2d_1 d_2 cos\varphi,$$
$$d_1 d_2 = \frac{a^2 - b^2 + c^2 - d^2}{2cos\varphi}$$

The formula for the area of a quadrilateral inscribed in a circle is as follows

$$S = \frac{1}{2} d_1 d_2 sin\varphi = \sqrt{(p-a)(p-b)(p-c)(p-d)}, \qquad p = \frac{a+b+c+d}{2}.$$
$$d_1 d_2 = \frac{2\sqrt{(p-a)(p-b)(p-c)(p-d)}}{sin\varphi},$$
$$S = \frac{1}{4}\sqrt{(b+c+d-a)(a+c+d-b)(a+b+d-c)(a+b+c-d)},$$
$$16S^2 = ((c+d)^2 - (a-b)^2)((a+b)^2 - (c-d)^2),$$
$$a^2 - b^2 + c^2 - d^2 = \frac{4S}{sin\varphi} cos\varphi,$$
$$ctg\varphi = \frac{a^2 - b^2 + c^2 - d^2}{4S},$$
$$\mathrm{ct}^2\, g\varphi + 1 = \frac{(a^2 - b^2 + c^2 - d^2)^2}{16S^2} + 1,$$
$$\frac{1}{\sin^2\varphi} = \frac{(a^2 - b^2 + c^2 - d^2)^2 + 16S^2}{16S^2},$$
$$sin\varphi = \frac{4S}{\sqrt{(a^2 - b^2 + c^2 - d^2)^2 + 16S^2}} =$$
$$= \frac{4S}{\sqrt{(a^2 - b^2 + c^2 - d^2)^2 + ((c+d)^2 - (a-b)^2)((a+b)^2 - (c-d)^2)}}$$
$$= \frac{2S}{(ac+bd)} = \frac{2S}{d_1 d_2} \Rightarrow d_1 d_2 = ac + bd.$$

Method 2. According to the circumscribed circle rule for a quadrilateral**:** $< B + < D = 180^0$ Using the equation and the Law of Cosines, we find the diagonal AC (Figure 2).

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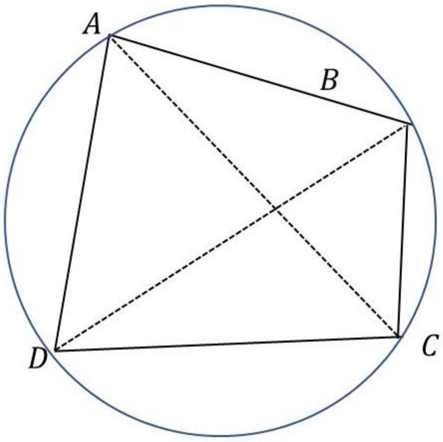
Figure 2.



$$d_1^2 = a^2 + b^2 - 2abcosB,$$

$$d_1^2 = c^2 + d^2 + 2cdcosB,$$

$$\begin{cases} d_1^2 - a^2 - b^2 = -2abcosB \\ d_1^2 - c^2 - d^2 = 2cdcosB \end{cases},$$

$$\begin{cases} cd(d_1^2 - a^2 - b^2) = -2abcdcosB \\ ab(d_1^2 - c^2 - d^2) = 2abcdcosB \end{cases}.$$

By combining the system, we obtain the following equation

$$cd(d_1^2 - a^2 - b^2) + ab(d_1^2 - c^2 - d^2) = 0,$$

$$d_1^2 = \frac{(ac+bd)(ad+bc)}{ab+cd}.$$

Using the same method, the second diagonal can be found from angles A and C; its proof is left to the reader.

$$d_2^2 = \frac{(ab+cd)(ac+bd)}{bc+ad};$$

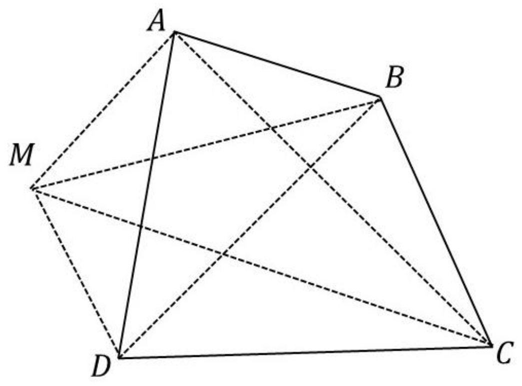
tWe multiply the squares of the found diagonals

$$d_1^2 d_2^2 = \frac{(ac+bd)(ad+bc)}{ab+cd} \cdot \frac{(ab+cd)(ac+bd)}{bc+ad}$$

$$d_1^2 d_2^2 = (ac+bd)^2 \Rightarrow d_1 d_2 = ac+bd.$$

Method 3. According to Mena's theorem, if ACACAC and BDBDBD are the diagonals of a quadrilateral and MMM is any point in the plane of the quadrilateral, the following equation holds (Figure 3)**.**

Figure 3.



$$\frac{AC}{BD} = \frac{BC \cdot CD \cdot AM^2 + AB \cdot AD \cdot CM^2}{AD \cdot CD \cdot BM^2 + AB \cdot BC \cdot DM^2}.$$

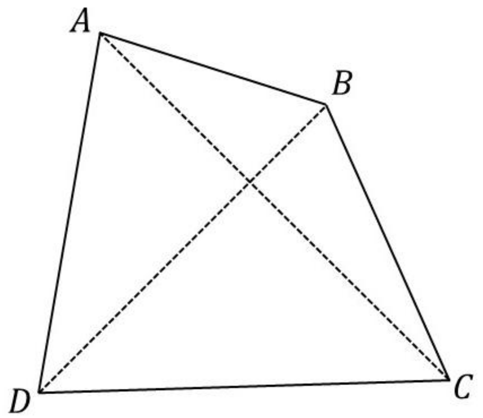
Using this equation, we consider the case where points M and A coincide [3].

In this case, the formula takes the following form:

$$\frac{AC}{BD} = \frac{BC \cdot CD \cdot AA^2 + AB \cdot AD \cdot CA^2}{AD \cdot CD \cdot BA^2 + AB \cdot BC \cdot DA^2} = \frac{AB \cdot AD \cdot CA^2}{AB \cdot AD(AB \cdot CD + BC \cdot AD)};$$
$$AC \cdot BD = AB \cdot CD + BC \cdot AD.$$

Method 4. According to Bretschneider's theorem, the following equation holds for a convex quadrilateral (Figure 4).

Figure 4.



$$(AC \cdot BD)^2 = (AB \cdot CD)^2 + (BC \cdot AD)^2 - 2AB \cdot CD \cdot BC \cdot AD \cdot \cos(\angle A + \angle C)$$

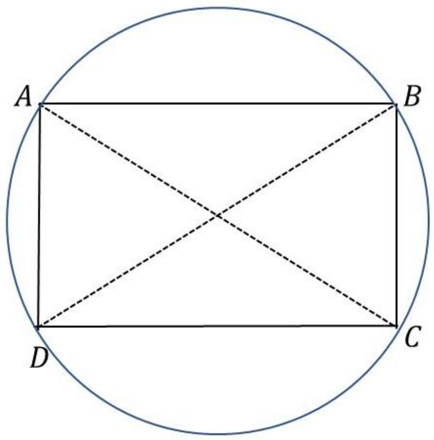
Considering a cyclic quadrilateral as a special case of this theorem, $\angle A + \angle C = 180^0$ the equation becomes known to us. $\cos(\angle A + \angle C) = -1$ Hence, the above formula becomes a complete square.

$$(AC \cdot BD)^2 = (AB \cdot CD + BC \cdot AD)^2 \Longrightarrow AC \cdot BD = B \cdot CD + BC \cdot AD.$$

Analysis and Results. By means of Ptolemy’s theorem, it is possible to prove several other theorems and solve complex problems. Below, we examine some of the outcomes of this theorem.

Case 1. Pythagorean Theorem. The sum of the squares of the legs is equal to the square of the hypotenuse (Figure 5).

Figure 5.

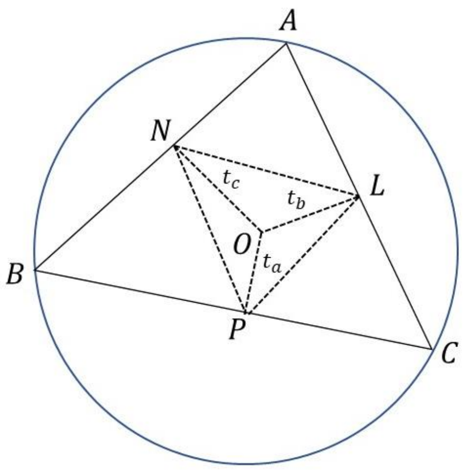


According to Ptolemy's theorem, $AC \cdot BD = AD \cdot BC + AB \cdot DC.$ the equality holds. In a rectangle, $AC = BD, AD = BC, AB = DC,$ and Since these equalities hold, the following result is obtained [2].

$$AC^2 = AD^2 + DC^2.$$

Case 2 (Figure 6). Carnot's Theorem. In any triangle, the sum of the distances from the center of the circumscribed circle to the sides of the triangle is equal to the sum of the radii of the inscribed and escribed circles.

Figure 6.



Proof. In triangle ΔABC, let O be the center of the circumcircle, R the circumradius, r the inradius, and denote the distances from O to the sides of the triangle by tat_ata, tbt_btb, and tct_ctc. According to the hypothesis of the theorem, the following equality holds:

$$t_a + t_b + t_c = R + r.$$

According to Ptolemy's theorem, the following equality holds in quadrilateral $ANOL$

$$AO \cdot NL = ON \cdot AL + OL \cdot AN, \qquad R \cdot \frac{a}{2} = t_c \cdot \frac{b}{2} + t_b \cdot \frac{c}{2}.$$

Similarly, from quadrilaterals $BNOP$ and $CLOP$, the following results are obtained.

$$R\cdot\frac{b}{2}=t_a\cdot\frac{c}{2}+t_c\cdot\frac{a}{2};\ \ R\cdot\frac{c}{2}=t_b\cdot\frac{a}{2}+t_a\cdot\frac{b}{2}.$$

If we add the obtained equalities term by term

$$R\left(\frac{a+b+c}{2}\right)=t_a\cdot\frac{b+c}{2}+t_b\cdot\frac{a+c}{2}+t_c\cdot\frac{a+b}{2},$$

$$Rp=(t_a+\ t_b+t_c)p-S=(t_a+\ t_b+t_c)p-pr,$$

$$t_a+\ t_b+t_c=R+r.$$

Conclusion and Recommendations. In conclusion, it can be stated that the proof methods of Ptolemy's theorem presented above encourage those interested in olympiad problems to approach the given problems creatively.

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