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Homotopy and Homology in Knot Theory

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Abstract: This paper provides a comprehensive investigation into the role of homotopy and homology within the framework of knot theory. The relevance of these concepts is underscored not only by their foundational importance in algebraic topology but also by their broad applicability across multiple scientific domains, including physics, biology, and computer science. In the era of globalization and the increasing necessity for interdisciplinary approaches to complex scientific challenges, a deep understanding of homotopy and homology in the context of knot theory is of particular significance.

Keywords: homotopy, homology, knot theory, invariance, isotopy, automorphism, homeomorphism, surfaces, homotopy invariants, homological invariants, polynomial invariant.

Introduction

In order to further improve the system of teaching mathematical sciences at all levels of education, to support the effective work of teachers, to expand the scope and enhance the practical relevance of scientific research, to strengthen ties with the international community, and to fulfill the objectives outlined in the State Program for the implementation of the Strategy of Action on five priority areas of development of the Republic of Uzbekistan for 2017–2021, during the "Year of the Development of Science, Education, and the Digital Economy":

A system is being created in Uzbekistan to foster mathematical thinking from an early age, in which the Institute of Mathematics will coordinate the teaching of mathematics in preschool educational institutions, schools, and universities.

“It is impossible to accelerate the development of mathematics using outdated teaching methods. Therefore, it is necessary to retrain teachers by creating educational programs based on well-established international practices. The methodology should be such that it inspires a love for mathematics in children. Students must understand that this science is essential in life and in every field,” emphasized Shavkat Mirziyoyev.

The relevance of homotopy and homology in knot theory lies not only in their mathematical importance but also in their wide range of applications across various fields of science such as physics, biology, and computer science. In the context of globalization and interdisciplinary approaches to solving scientific problems, understanding these concepts becomes particularly significant.

This article aims to address problems related to the computation of homotope and homology invariants of knots, as well as to explore their interconnections and applications in various contexts.

Knot theory is one of the most fascinating and actively developing areas of topology, studying the properties and relationships of different knots in three-dimensional space. Knots, as mathematical objects, are closed curves that can be deformed into one another without cutting or intersecting. Knot-related questions have deep roots in various areas of mathematics and even in physics, where knots can model phenomena such as magnetic fields or molecular biology structures. In recent decades, interest in knots has grown due to their applications in fields like quantum physics, biology, and computer science, making the study of knots especially relevant.

One of the key aspects of knot theory is the understanding of their topological properties, which are independent of how they are specifically represented in space. For this purpose, mathematicians use various tools and methods, including homotopy and homology. Homotopy, as a concept, allows the study of continuous deformations of functions and spaces, while homology provides powerful algebraic tools for examining the topological properties of spaces. These two branches of mathematics, being interrelated, have a significant impact on knot theory, enabling the formalization and systematization of knowledge about various classes of knots, their properties, and interrelations.

The basic concepts of homotopy and homology form the foundation for the further study of knots. Homotopy, in particular, examines the notion of homotopic equivalence, which allows the classification of knots based on their ability to be continuously transformed into each other. This leads to the creation of different classes of knots that share the same topological properties. Homology, in turn, provides tools for calculating knot invariants, such as homology groups, which help to distinguish between knots that cannot be deformed into one another. These invariants become the main subject of study in knot theory, allowing researchers to discover new patterns and relationships.

Homotopic equivalence of knots is a subject that deserves special attention. It reveals how knots can be related through continuous transformations and how these transformations affect their properties. It is important to note that not all knots are homotopic ally equivalent, and the existence of homotop invariants allows us to identify knots with unique characteristics. This forms the basis for a deeper analysis of knots and their interrelations, which is a crucial step in understanding their structure and properties.

The **knot group**, as an algebraic structure, also plays a key role in knot theory. It allows for the formalization of operations on knots and the study of their properties in the context of homotopy and homology. The study of the knot group opens new horizons for understanding their structure and the relationships between different classes of knots. These groups can be used to classify knots and identify their invariants, which is a vital aspect of knot theory.

Applications of homotop and homology to knot theory represent the final stage of our investigation. They demonstrate how theoretical concepts can be applied to practical problems related to knots. For example, homology methods can be used to compute knot invariants that have significant value in various scientific and technological fields. These applications highlight the connection between abstract mathematics and real-world problems, making the study of knots even more meaningful.

Homotopy and homology are two fundamental concepts in algebraic topology that play an essential role in understanding and classifying topological spaces, as well as in studying various objects such as knots. These concepts allow us to investigate the properties of spaces that are invariant under continuous deformations and help to reveal structural aspects that may not be immediately obvious. In the context of knot theory, homotopy and homology provide powerful tools for analyzing and classifying knot structures and for understanding their interconnections and properties.

Homotopy is a concept that describes how two continuous functions, mapping the same space, can be "connected" through continuous deformations. Formally, if we have two continuous maps $f, g: X \to Y$ from a topological space X to a topological space Y, we say that f and g are homotopic if there exists a continuous family of maps $H: X \times [0, 1] \to Y$, such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

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This map H is called a homotopy between f and g, and the parameter $t \in [0, 1]$ is interpreted as the "time" of deformation. Thus, $H(x, t)$ represents a continuous family of mappings from X to Y, continuously transforming f into g.

Homotopy defines an **equivalence relation** on the set of continuous maps $\mathbf{X} \to \mathbf{Y}$, since it satisfies the following conditions:

- ✓ **Reflexivity**: $f \simeq f$
- ✓ **Symmetry**: if $f \simeq g$, then $g \simeq f$
- ✓ **Transitivity**: if $f \simeq g$ and $g \simeq h$, then $f \simeq h$

Therefore, mappings can be classified by **homotopy classes**, where all functions within a class can be continuously deformed into each other. This allows us to study topological properties of spaces at a more abstract level without fixing specific maps.

Homotopy also plays an important role in studying properties such as connectedness and simple connectivity. For example, if a space X is connected, then any continuous map $f: X \to \{y_0\} \subset Y$, where $\{y_0\}$ is a single-point space, is homotopic to any other such map. This property underlies many results in algebraic topology, including theorems on homotopy equivalence of spaces.

Homology, on the other hand, is a more complex concept related to studying the "cyclic" properties of topological spaces. Homology groups provide algebraic invariants that describe how a space can be divided into "pieces" and how these pieces are connected to one another. The main idea of homology is to consider chains, which are combinations of simplices (or more general objects) in the space, and to determine which of these chains are boundaries of other chains. Chains and boundaries form the fundamental elements of homological theory.

Let X be a topological space. Denote by $C_n(X)$ the abelian group of n-chains, i.e., the free abelian group generated by singular n-simplices in X:

$$C_n(X) = \mathbb{Z}[\mathrm{Sing}_n(X)]$$

Define the boundary operator

$$\partial_n: C_n(X) \to C_{n-1}(X)$$

satisfying the condition $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$, i.e., $\partial^2 = 0$

The n-th cycle group is defined as the kernel of the boundary operator:

$$Z_n(X) = \ker(\partial_n) \subseteq C_n(X)$$

The n-th boundary group is the image of the boundary operator from the next dimension:

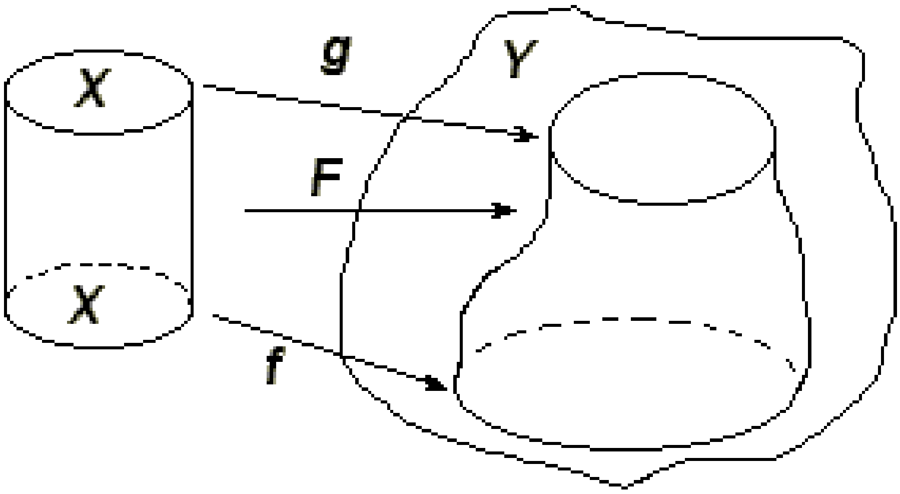
$$B_n(X) = \mathrm{im}(\partial_{n+1}) \subseteq C_n(X)$$

The n-th homology group is then defined as the quotient group:

$$H_n(X) = Z_n(X) / B_n(X)$$

The groups $H_n(X)$ are invariant under homotopy equivalence and serve as algebraic invariants reflecting the topological properties of the space X, particularly its **cyclic** structure.

In knot theory, homotopy and homology play a key role in understanding knot structures and their relationships. Knots can be considered as embeddings of a circle into three-dimensional space, and different knots may be related through homotopic transformations. Homotopy allows us to study how knots.



PROPERTIES OF HOMOTOPY

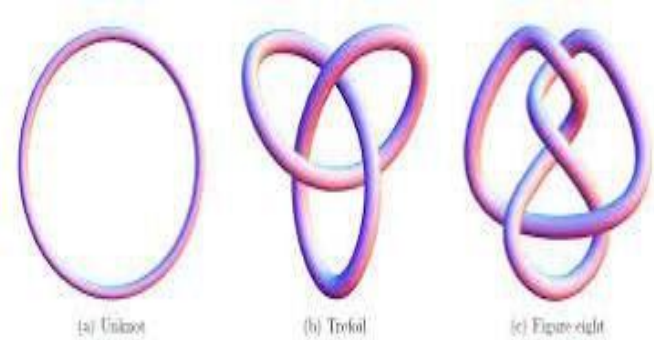
Isotopy invariance.

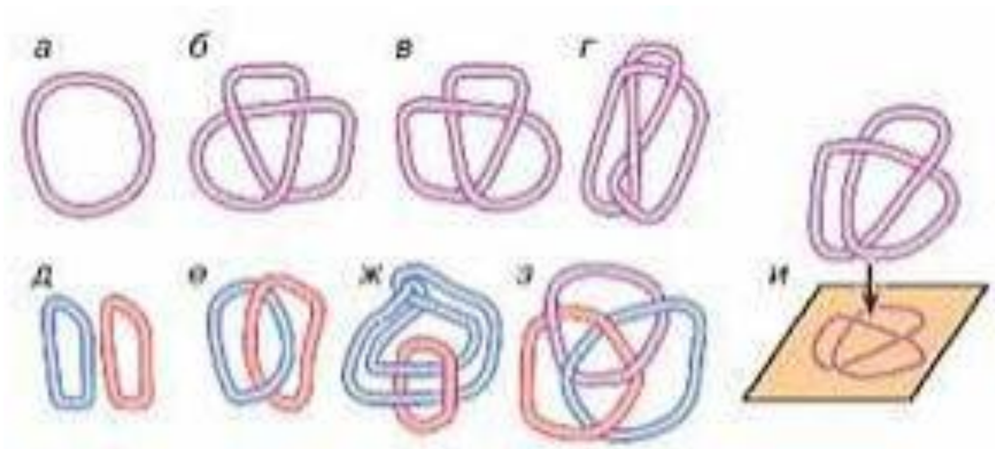
Theorem: If knots K_1 and K_2 are isotopic in S^3, then their complements are homeomorphic, and therefore $\pi_1(S^3 \setminus K_1) \cong \pi_1(S^3 \setminus K_2)$.

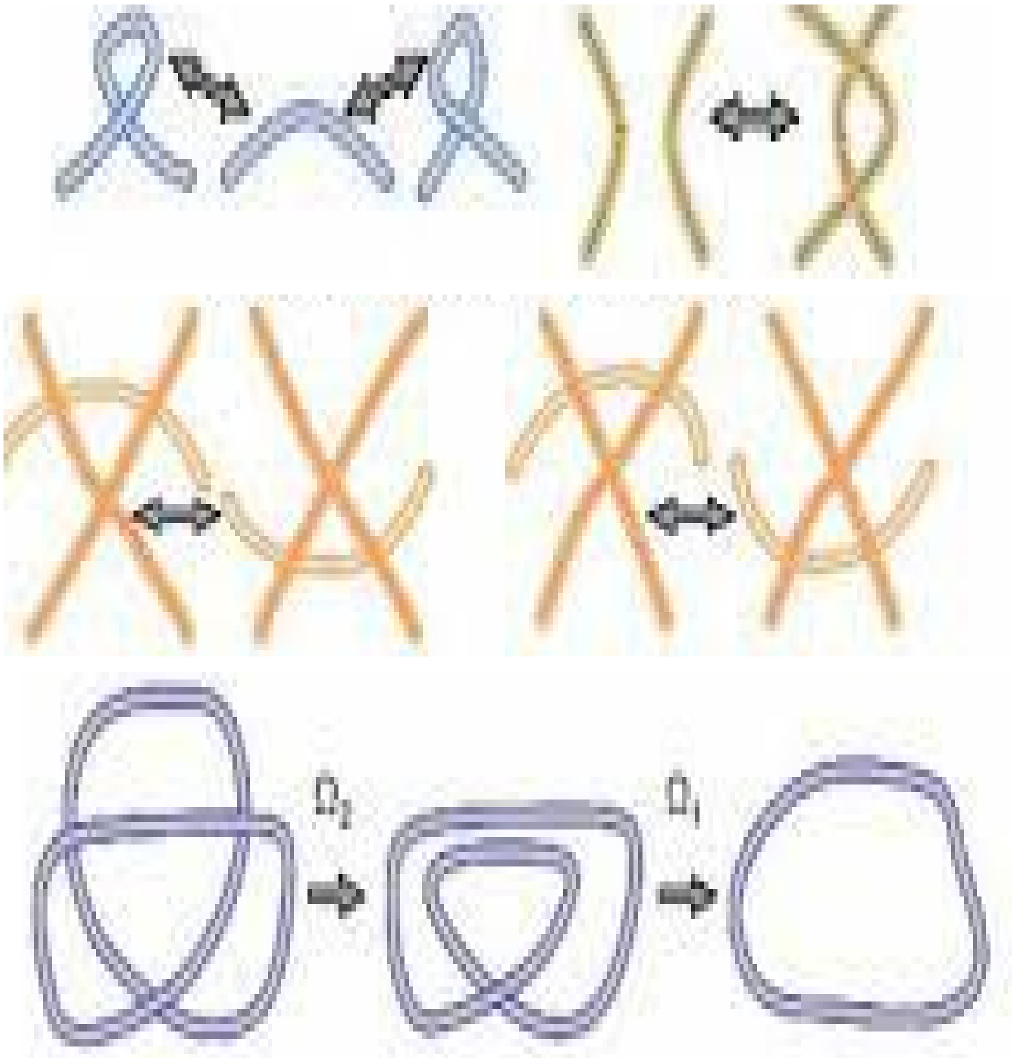
(Waldhausen, 1968): If two non-trivial prime knots have isomorphic knot groups (taking into account the peripheral structure), then these knots are isotopic.

Corollary: The knot group with a peripheral structure (for example, meridian and longitude) completely determines the knot up to isotopy.

These elements are included in the description of the "peripheral subgroup" of the knot group. Without them, different knots (for example, the trivial and mirror knots) can have isomorphic fundamental groups.







HOMOTOPY PROPERTIES

The homotopy relation is an equivalence relation on the set of continuous maps X→Y: it is reflexive (f ≃ f), symmetric, and transitive. In particular, if f ≃ g and g ≃ h, then f ≃ h.

Homotopy is compatible with composition: if f1 ≃ g1 : X→Y and f2 ≃ g2 : Y→Z, then f2∘f1 ≃ g2∘g1. It follows that all constructive properties of mappings are preserved under homotopic transformation (for example, induced homomorphisms in fundamental and homology groups coincide).

Homotopy equivalence of spaces. Spaces X and Y are homotopy equivalent (having the same homotopy type) if there exist f : X → Y and g : Y → X such that g∘f ≃ idX and f ∘g ≃ idY. In this case, f and g are said to be mutually inverse from the point of view of homotopy.

Any homeomorphism is a special case of homotopy equivalence.

Contractibility. A space X is called contractible if the identity map idX is homotopic to a constant map. Then X is homotopy equivalent to a point, and all mappings from X to any space "slide" into one point. In particular, any two mappings from a contractible space to an arbitrary Y are homotopic to each other.

Thus, homotopy formulates the concept of continuous deformability of mappings without breaks, and the homotopy relation has standard equivalence properties.

DEFINITION OF HOMOLOGY

Let a chain complex of abelian groups be given:

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$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots, \qquad \partial_n \circ \partial_{n+1} = 0.$$

Denote Z_n= ker ∂_n as the group of n-cycles (elements without a boundary) and B_n=Im ∂_{n+1} as the group of n-boundaries. Since $\partial_n \partial_{n+1}=0$, the inclusion $B_n \subseteq Z_n$.

holds.

The n-th homology of this complex is defined as the quotient group:

$H_n= Z_n / B_n$,

Its elements are called homology classes of n-cycles. Homologies serve as an algebraic measurement of how much the complex is not exact.

The word "exact" (from English "exact") is a term from homological algebra, which means that the chain complex (or sequence of homomorphisms) has the property that the image of the previous map coincides with the kernel of the next one.

In particular, for a topological space X, a singular chain complex $(C_n(X), \partial)$ is constructed: the group $C_n(X)$ is a free abelian group generated by all continuous maps $\sigma : \Delta^n \to X$ from the standard n-simplex to X. The boundary operator

$\partial_n : C_n(X) \to C_{n-1}(X)$ is defined by the image of the boundary faces of the simplex (with signs). Then the n-th homologies of the space X are defined as

$H_n(X)= Z_n(X) / B_n(X)$. Formulation: The singular homology of the space X is the homology of the chain complex $(C_*(X), \partial)$.

Properties of Homology

Functoriality. A continuous map f : X → Y induces homomorphisms of chain groups $f_* C_n(X) \to C_n(Y)$ f_* (mapping simplices via f) and, as a consequence of

$\partial f_* = f_* \partial$, a homomorphism of homologies.

KNOT CLASSIFICATION

Definition 2.1.1 (Knot). Let S^1 denote the unit circle, and R^3 the three-dimensional Euclidean space. A knot is a smooth embedding:

$K: S^1 \to R^3$,

considered up to smooth reparamet rization. We identify the embedding K with its image $K(S^1) \subset R^3$, assuming that this image is a simple closed smooth curve in R^3

Definition 2.1.2 (Knot equivalence). Let $K_1, K_2: S^1 \to R^3$ be two knots. K_1 and K_2 are said to be equivalent as knots if there exists a diffeomorphism

$f : R^3 \to R^3$,

such that:

1. $f (K_1(S^1))= K_2 (S^1)$;
2. f preserves the orientation of R^3, that is, $\det (D f_x) > 0$ for all $x \in R^3$, if f is smooth, or f induces the identity map on the orientation bundle of R^3 (in the topological category);
3. If necessary (in the case of oriented knots) f also preserves the orientation along the curve $K_1 (S^1)$.

Such a diffeomorphism f is called an orientation-preserving automorphism of space, and its existence means that the knots K_1 and K_2 have the same topological structure up to orientation.

Definition 2.1.3. (Isotopy of knots).

Smooth knots $K_1, K_2 : S^1 \to R^3$ are called isotopic if there exists a smooth mapping:

$F : S^1 \times [\ 0{:}1\ ] \to R^3$, such that:

1. For each $t \in [0, 1]$, the mapping $F_t(\cdot) := F_t(\cdot,) : S^1 \to R^3$ is an embedding;
2. $F_0 = K_1, F_1 = K_2$.

In this case, we say that $\{ F_t \}_{t \in [\ 0:1\ ]}$ is a smooth isotopy deforming the knot K_1 into the knot K_2in a continuous manner.

Isotopic knots are, obviously, equivalent; however, there exists a broader notion of equivalence through an orientation-preserving homeomorphism of space, which does not always reduce to isotopy.

Definition 2.1.4 (Knot classification). The classification of knots consists in determining all equivalence classes of embedding $S^1 \to R^3$ up to orientation-preserving automorphisms of R^3. Two knots belong to the same class if there exists an orientation-preserving homeomorphism $f : R^3 \to R^3$ f that maps one knot to the other.

Example 2.1. The trivial knot is a knot equivalent to the standard circle in R^3, that is the image of the embedding:

$K_0(\theta) = (\cos\theta, \sin\theta, 0), \theta \in [\ 0{:}\ 2\pi\ ]$.

Any knot isotopic to K_0 is considered trivial.

Conclusion

The complete classification of knots is an extremely complex task, as there are infinitely many knots (for example, there exists a countable number of non-equivalent torus knots Tp,qT_{p,q}Tp,q). Classification involves the study of invariants such as the fundamental group of the knot complement, homotopy and homology invariants, polynomial invariants, and others.

SUMMARY

The conclusion of this article on the topic "Homotopy and Homology in Knot Theory" represents a generalization of the aspects studied and the findings obtained through the analysis of key concepts related to homotopy and homology, their application in the context of knot theory, as well as the examination of homotopic equivalence of knots and the knot group. Throughout the work, it was established that homotopy and homology, as important tools of algebraic topology, open new horizons for understanding the structure and properties of knots.

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