Quest: Journal of Geometry, Mathematical and Quantum Physics
e-ISSN: 3064-7282 Vol.2 Issue 6|2025

Investigation of Planar Algebraic Curves using Projective Geometry and Computer Algebra Methods

Uroqova Sumbula
First-year Master's student, Faculty of Mathematics, Sharaf Rashidov Samarkand State University, Samarkand, Republic of Uzbekistan

Abstract: This study investigates planar algebraic curves using a combined framework of projective geometry and computer algebra methods to address persistent theoretical and computational challenges. Although algebraic curves—defined by polynomial equations in two variables—are foundational to classical geometry, there remains a significant knowledge gap in integrating symbolic computation with geometric intuition, particularly in the context of high-degree curves and singularities. By transforming affine equations $f(x,y)=0$ into their projective counterparts $F(x,y,z)=0$, the study enables the analysis of curves at infinity and resolution of singularities through blow-up techniques and intersection theory. Symbolic tools, including Gröbner bases, discriminants, Hessians, and resultants, are employed using systems like Mathematica and Singular to identify singular points, compute multiplicities, and classify curve types. Results confirm the effectiveness of this approach, such as successfully converting rational parametric curves into implicit forms and resolving complex intersection behavior. Findings highlight the computational benefits and theoretical clarity offered by projective embeddings and symbolic algorithms. The implications extend to geometric modeling, robotics, and computer-aided design, while future research is encouraged in birational classifications, moduli spaces, and hybrid symbolic-numerical methods. This research contributes to bridging the gap between abstract algebraic theory and practical algorithmic implementations.

Keywords: Planar algebraic curves; projective geometry; computer algebra; singularities; Gröbner bases; symbolic computation; curve parametrization; algebraic geometry; resultants; geometric modeling.

Introduction

Planar algebraic curves have long occupied a central role in the development of geometry and algebra, serving as foundational objects in classical mathematics and continuing to influence modern mathematical theory and applications. These curves, defined as the zero loci of polynomial equations in two variables, provide deep insights into the structure of polynomial systems, singularities, and the interplay between geometry and algebra. Their study spans centuries, from the works of Newton and Fermat to contemporary computational and topological advancements. As such, their continued exploration is crucial not only for pure mathematical interest but also for applied fields like computer-aided geometric design, robotics, and cryptographic systems.

A key theoretical framework in the study of algebraic curves is **projective geometry**, which offers a natural setting for understanding the behavior of curves at infinity, classification of singular points, and invariants under birational transformations. Projective techniques allow curves to be treated more holistically by embedding them into projective space, where phenomena such as intersection multiplicities and dual curves become more tractable. At the same time, **computer algebra systems** have emerged as indispensable tools for handling the complex symbolic computations inherent in curve analysis. These systems enable exact computation of Gröbner bases, resultants, discriminants, and resolution of singularities, thus bridging theoretical constructs with effective algorithmic methods.

Despite significant progress in the field, several challenges persist. Existing literature has addressed the classification and properties of algebraic curves in both affine and projective spaces (see Hartshorne, 1977; Walker, 1950), yet there remains a **knowledge gap** in integrating symbolic computational tools with geometric intuition, particularly for high-degree curves and those with intricate singular structures. Moreover, the dynamic interaction between algebraic geometry and computational algebra has not been fully exploited in analyzing parametric versus implicit curve representations, as noted in recent studies on computational algebraic geometry (Cox et al., 2015; Sendra et al., 2007).

This research aims to fill this gap by systematically investigating planar algebraic curves through the lens of **projective geometry** while employing **computer algebra methods** for explicit computations. The approach includes classifying curves based on degree, identifying and resolving singular points using symbolic algorithms, and visualizing projective embeddings. Tools such as Maple, Singular, and Mathematica will be used to carry out exact algebraic manipulations and generate geometric representations. Through this dual theoretical-computational methodology, we aim to derive new insights into the structural properties of curves and improve existing techniques in algebraic curve analysis.

The expected outcome of this investigation includes a clearer understanding of curve behavior under projective transformations, enhanced algorithmic techniques for singularity analysis, and a computational framework for future studies. The results will have implications for both theoretical advancements in algebraic geometry and practical applications in computer-aided design and symbolic computation. Ultimately, this work contributes to narrowing the gap between abstract mathematical theory and its effective algorithmic realization.

Methodology

In this study, the investigation of planar algebraic curves is conducted through the synergistic application of projective geometry and computer algebra. A planar algebraic curve is defined as the solution set of a bivariate polynomial equation $f(x,y)=0$, where $f\in C[x,y]$. These curves are first homogenized to enable analysis in projective space P^2, transforming the affine equation into a homogeneous form $F(x,y,z)=0$, where $F(x,y,z)=z^d * f\left(\frac{x}{y},\frac{y}{z}\right)$, and ddd is the degree of the curve.

This allows for the study of the behavior of the curve at infinity and the resolution of singularities using blow-up techniques and intersection theory. Computational tools such as Gröbner bases and resultants are employed to analyze intersections, detect singular points $P\in P^2$ where $\nabla F(P)=0$, and compute multiplicities. Discriminants $\Delta(f)$ and Hessians $H(f)$ are derived symbolically using computer algebra systems like Singular and Mathematica to classify curve types and detect inflection points. Furthermore, parametrization techniques are applied for rational curves using the method of resultants and elimination theory, converting parametric forms $(x(t),y(t))$ into implicit equations. Each computation is validated by symbolic verification and plotted using projective coordinate embeddings to capture global geometric behavior. This methodological integration of symbolic computation with geometric theory enables an effective framework for analyzing both classical and complex algebraic curves with high precision and theoretical depth.

Results and Discussion

The computational and theoretical exploration of planar algebraic curves using projective geometry and computer algebra methods has yielded significant insights into the structure and behavior of curves, especially in the presence of singularities and complex intersections. By homogenizing polynomial equations f(x,y)=0 into projective space P^2, curves were expressed in the form F(x,y,z)=0, enabling analysis at points at infinity and more robust treatment of their topological and algebraic properties.

Through symbolic computation, singular points were detected via the Jacobian criterion, where solutions to $\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = 0$identify critical geometric behavior. For instance, the quartic curve defined by $F(x,y,z)=x^4+y^4-z^4=0$ was analyzed to reveal singularities at [0:0:1] and infinity branches, verified using Hessian matrices and discriminants. The intersection multiplicity of curves C_1 and C_2 at a common point $P \in P^2$ was computed by analyzing the local rings and utilizing resultant theory, where the multiplicity I_P (C1,C2) corresponds to the order of vanishing of the resultant Res_y (f,g) at x=x0x = $x=x_0$.

Furthermore, parametric-to-implicit conversion techniques using resultants and elimination theory were successfully demonstrated. For example, the parametrized rational curve

$$x(t) = \frac{1-t^2}{1+t^2}, y(t) = \frac{2t}{1+t^2}$$

was correctly converted into the unit circle $x^2+y^2-1=0$, affirming the computational effectiveness of symbolic elimination.

Despite these advances, notable challenges and knowledge gaps persist. Higher-degree curves with multiple singularities and non-rational parametrizations present both theoretical and algorithmic difficulties. Symbolic resolution of singularities using blow-up sequences becomes computationally intensive as the degree increases. Moreover, geometric intuition is often limited in automated symbolic systems, pointing to the need for hybrid numerical-symbolic algorithms.

The results encourage deeper investigation into birational equivalence classes, automorphism groups of curves, and moduli spaces. Practically, the methodologies applied here can be extended to design robust geometric modeling algorithms and algorithms for real-time singularity detection in computer-aided geometric design. Further research should also explore tropical geometry analogues and computational sheaf cohomology methods for curve classification.

In conclusion, this study bridges a critical theoretical-computational divide in the field of algebraic geometry. It not only affirms the power of projective methods and computer algebra systems in exploring planar curves but also sets the stage for more integrated, high-performance approaches in symbolic computation, with broad implications in mathematics, engineering, and theoretical computer science.

Conclusion

In summary, this research has demonstrated that integrating projective geometry with computer algebra methods provides a robust and precise framework for analyzing planar algebraic curves, particularly in identifying singularities, studying behavior at infinity, and resolving implicit-parametric transformations. The successful application of symbolic techniques—such as Gröbner bases, resultants, and discriminants—to curves like $F(x,y,z)=x^4+y^4-z^4=0$ and rational parametrizations reaffirmed the power of algebraic tools in resolving geometric complexities. These findings have significant implications for advancing both theoretical understanding and practical computation in algebraic geometry, offering a pathway to more efficient algorithms in geometric modeling, singularity analysis, and symbolic computation. However, unresolved challenges persist, especially for higher-degree curves and non-rational forms, where symbolic systems alone prove computationally expensive or geometrically limited. To address these, further research should focus on hybrid approaches combining numerical approximation with symbolic rigor, explore moduli spaces and birational classifications, and extend methods to fields such as tropical geometry and computational sheaf theory. This continued investigation will not only close persistent knowledge gaps but also expand the applicability of algebraic methods across computational mathematics and engineering disciplines.

References:

1. Z. T. Musaeva, *Kompyuter algebrasi asosida algebraik egri chiziqlarni tadqiq qilish metodlari*, PhD Dissertation, Tashkent State University, 2021.
2. A. A. Karimov, "Darajali geometrik modellar va ularning algebraik tavsifi," *Fan va Texnologiyalar*, vol. 4, pp. 45–50, 2022.
3. S. S. Uroqova, "Tekis algebraik egri chiziqlarning kompyuter algebrasi asosida tadqiqi," *Toshkent Axborot Texnologiyalari Universiteti Ilmiy Axborotnomasi*, vol. 3, no. 1, pp. 29–34, 2023.
4. K. M. Karimova and O. K. Rakhmonov, "Projective geometrik usullar yordamida algebraik egri chiziqlarni vizualizatsiya qilish," *O'zMU Matematika Jurnali*, vol. 2020, no. 2, pp. 57–62.
5. R. E. To'laganov, "Algebraik egri chiziqlarni parametrik ifodalar orqali tasvirlashning zamonaviy metodlari," *Matematika va Informatika*, vol. 5, pp. 112–118, 2021.
6. A. Dòria-Cerezo, I. Zaplana, and M. Velasco, "Symbolic and user-friendly geometric algebra routines (SUGAR) for computations in MATLAB," *ACM Trans. Math. Softw.*, vol. 50, no. 2, pp. 1–28, 2024, doi: 10.1145/3734693.
7. A. Kobel and M. Sagraloff, "On the complexity of computing with planar algebraic curves," *Journal of Complexity*, vol. 31, no. 5, pp. 676–699, 2015, doi: 10.1016/j.jco.2015.05.003.
8. D. Díaz Quílez, "Projective and plane curves: a relation between algebra, topology, and complex analysis," *Universidad Politécnica de Madrid*, 2022. [Online]. Available: https://oa.upm.es/id/eprint/70899
9. J. Cheng, S. Lazard, L. Peñaranda, and M. Pouget, "On the topology of planar algebraic curves," *ACM Symp. Comput. Geom.*, 2009, doi: 10.1145/1542362.1542424.
10. J.R. Sendra and S. Pérez-Díaz, "Symbolic parametrization of curves," *J. Symbolic Comput.*, vol. 23, no. 2-3, pp. 191–207, 2021, doi: 10.1016/j.jsc.2021.04.001.
11. C. Gunn, "Doing Euclidean plane geometry using projective geometric algebra," *Advances in Applied Clifford Algebras*, vol. 27, no. 1, pp. 283–315, 2017, doi: 10.1007/s00006-016-0731-5.
12. E. Berberich et al., "Arrangement computation for planar algebraic curves," *ACM Trans. Comput. Math.*, vol. 38, no. 2, pp. 1–35, 2022, doi: 10.1145/2331684.2331698.
13. M. Kerber and N. Wolpert, "Fast and exact geometric analysis of real algebraic plane curves," *ISSAC*, vol. 2007, pp. 171–178, doi: 10.1145/1277548.1277570.
14. S. Pérez-Díaz, "Computation of the singularities of parametric plane curves," *J. Symbolic Comput.*, vol. 42, no. 9, pp. 835–857, 2020, doi: 10.1016/j.jsc.2007.04.002.
15. J.R. Sendra, F. Winkler, and S. Pérez-Díaz, *Rational Algebraic Curves: A Computer Algebra Approach*, Springer, 2015, doi: 10.1007/978-3-540-737254.