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Multiple Complex Variable Cauchy-Type Integral: Existence Condition

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Abstract: On the existence condition of the Koshi type integral in the domain of many complex variables.

$Let\ the\ function\ \varphi(t) = \varphi(t_1, t_2, \dots, t_n)$ be defined in the Δ-domain of the Anglo-American ball (framework).

Let us draw a polydisk $C(t_k, r_k)$ of any sufficiently small radius r_k centered at the point $t = (t_1, t_2, \dots, t_n) \in \Delta$ such that each contour $D_k^{\cdot}$ intersects with the polydisks at only two points. We denote the portion of the contour $D_k^{\cdot}$ inside the polydisk $C(t_k, r_k)$, by $d_k^{\cdot}$ and the rest as $D_k^{\cdot} - d_k^{\cdot}$. We define the product domain: $D_\varepsilon = (D_1^{\cdot} - d_k^{\cdot}) \times (D_2^{\cdot} - d_k^{\cdot}) \times \dots \times (D_n^{\cdot} - d_k^{\cdot}) \cdot$ It is clear that as, $\varepsilon \to 0$, $D_\varepsilon \to \Delta$.

Definition 1: If $\lim_{\varepsilon \to 0} \Phi_\varepsilon(t)$ where

$$\Phi_\varepsilon(t) = \frac{1}{(2\pi i)^n} \int_{D_\varepsilon} \frac{\varphi(\tau)}{\prod_{k=1}^{n} (\tau_k - z_k)} d\tau) \qquad (1)$$

$$\Phi(t) = \frac{1}{(2\pi i)^n} \int_{\Delta} \frac{\varphi(\tau)}{\prod_{k=1}^{n} (\tau_k - t_k)} d\tau \qquad (2)$$

exists, then we say that the special integral exists in the sense of the initial value of Cauchy, and this formation is referred to as:

$$\lim_{\varepsilon \to 0} \Phi_\varepsilon(t) = V \cdot P \frac{1}{(2\pi i)^n} \int_{\Delta} \frac{\varphi(\tau)}{\prod_{k=1}^{n} (\tau_k - t_k)} d\tau = V \cdot P\, \Phi(t)$$

We will call the initial part of the special integral (2) the simple integral and briefly denote it also as $\Phi(t) = \frac{1}{2^n} S\varphi(\tau)$.

(2) The initial value of the special integral does not always exist. Therefore, we will determine for which class of functions it is valid.

1-теорема. If the integrand of (2) satisfies the Hölder condition in the domain Δ, then the special integral (2) exists in the sense of the principal value.

Proof: To simplify notation, we prove the theorem for $n = 3$. In the proof, we use inequalities that satisfy the Holder condition.

(2) Let the integrand be: $\varphi(\tau_1, \tau_2, \tau_3)$

$\varphi(\tau_1, \tau_2, \tau_3) = \varphi_3(\tau; t) + \varphi_3\left(\tau_{t_1}; t\right) + \varphi_3\left(\tau_{t_2}; t\right) + \varphi_3\left(\tau_{t_3}; t\right) + +\varphi_3\left(t_{\tau_1}; t\right) +$

$+\varphi_3(t_{\tau_2};t)+\varphi_3(t_{\tau_3};t)+\varphi(t_1,t_2,t_3)$ Substituting this into the expression:

$$\Phi_\varepsilon(t)=\frac{1}{(2\pi i)^3}\int_{D_\varepsilon}\frac{\varphi_3(\tau,t)}{\prod_{k=1}^3(\tau_k-t_k)}d\tau+\frac{1}{(2\pi i)^3}\sum_{k=1}^3\int_{D_\varepsilon}\frac{\varphi_3(\tau_{t_k},t)}{\prod_{k=1}^3(\tau_k-t_k)}d\tau+$$

$$+\frac{1}{(2\pi i)^3}\sum_{k=1}^3\int_{D_\varepsilon}\frac{\varphi_3(t_{\tau_k},t)}{\prod_{k=1}^3(\tau_k-t_k)}d\tau+\frac{\varphi(t)}{(2\pi i)^3}\int_{D_\varepsilon}\frac{1}{\prod_{k=1}^3(\tau_k-t_k)}d\tau \qquad (3)$$

(3) To estimate

$\overset{3}{S}_\varepsilon\varphi_3(\tau,t)$; $\overset{2}{S}_\varepsilon\varphi_3(\tau_{t_k},t)$ $(k=1,2,3)$; $\overset{1}{S}_\varepsilon\varphi_3(t_{\tau_k},t)$ $(k=1,2,3)$ и $\overset{0}{S}_\varepsilon$. We use:

$\overset{3}{S}_\varepsilon\varphi_3(\tau,t)$, мы находим $|\varphi_3(\tau;t)|\le 4\prod_{k=1}^3 A_k^{\frac{1}{3}}|\tau_k-t_k|^{\frac{\alpha_k}{3}}$.

$$\left|\overset{3}{S}_\varepsilon\varphi_3(\tau,t)\right|\le\frac{4\sqrt[3]{A_1A_2A_3}}{(2\pi)^3}\int_{D_\varepsilon}\prod_{k=1}^3|\tau_k-t_k|^{\frac{\alpha_k}{3}-1}|d\tau_k|=$$

$$=\frac{1}{2\pi^3}\prod_{k=1}^3\sqrt[3]{A_k}\int_{D_k^{\cdot}-d_k^{\cdot}}|\tau_k-t_k|^{\frac{\alpha_k}{3}-1}|d\tau_k| \qquad (4)$$

Each integral on the right-hand side exists in the Riemann sense as в ε→0 For the remaining integrals, we apply:

$$\left|\frac{1}{2\pi i}\int_{D_k^{\cdot}-d_k^{\cdot}}\frac{d\tau_k}{\tau_k-t_k}\right|<1,\qquad k=1,2,3 \qquad (5)$$

Using this and bounds such as: $\overset{3}{S}_\varepsilon\varphi_3(\tau_{t_3},t)$ (5) and

$\left|\varphi_3\left(t_{\tau_p};t\right)\right|\le 2\prod_{k=1}^3 A_k^{\frac{1}{2}}|\tau_k-t_k|^{\frac{\alpha_k}{2}};\ p=\overline{1,3};\qquad k\ne p\ p=3.$

$$\left|\overset{3}{S}_\varepsilon\varphi_3(\tau_{t_3},t)\right|=\left|\frac{1}{2\pi i}\int_{D_k^{\cdot}-d_3^{\cdot}}\frac{d\tau_3}{\tau_3-t_3}\right|\left|\frac{1}{(2\pi i)^2}\int_{D_k^{\cdot}-d_1^{\cdot}}\int_{D_k^{\cdot}+d_2^{\cdot}}\frac{\varphi_3(\tau_{t_3},t)d\tau_1d\tau_2}{(\tau_1-t_1)(\tau_2-t_2)}\right|$$

$$\le\frac{1}{2\pi^2}\prod_{k=1}^2\sqrt{A_k}\int_{D_k^{\cdot}-d_k^{\cdot}}|\tau_k-t_k|^{\frac{\alpha_k}{2}-1}|d\tau_k| \qquad (6)$$

we find convergence of all terms as ε→0 in the simple Roman sense. Similarly, estimates will be appropriate for integrals $\overset{2}{S}_\varepsilon\varphi_3(\tau_{t_2},t)$ и $\overset{2}{S}_\varepsilon\varphi_3(\tau_{t_1},t)$

for $\overset{1}{S}_\varepsilon\varphi_3(t_{\tau_3},t)$we find (5) and $\left|\varphi_3\left(t_{\tau_p},t\right)\right|\le A_p|\tau_p-t_p|^{\alpha_p}$, $p=\overline{1,3}$, taking into account $p=3$.

$$\left|\overset{1}{S}_\varepsilon\varphi_3(t_{\tau_3},t)\right|=\left|\frac{1}{2\pi i}\int_{D_1^{\cdot}-d_1^{\cdot}}\frac{d\tau_1}{\tau_1-t_1}\right|\cdot\left|\frac{1}{2\pi i}\int_{D_2^{\cdot}-d_2^{\cdot}}\frac{d\tau_2}{\tau_2-t_2}\right|\cdot$$

$$\cdot\left|\frac{1}{2\pi i}\int_{D_3^{\cdot}-d_3^{\cdot}}\frac{\varphi_3(t_{\tau_3},t)d\tau_3}{\tau_3-t_3}\right|\le\frac{A_3}{2\pi}\int_{D_3^{\cdot}-d_3^{\cdot}}|\tau_3-t_3|^{\alpha_3-1}|d\tau_3|.$$

This implies the approximation $\overset{1}{S}_\varepsilon\varphi_3(t_{\tau_3},t)$ as ε→0.

It is also easy to see the convergence of $\overset{1}{S}_{\varepsilon}\varphi_3(t_{\tau_2},t)$ and $\overset{1}{S}_{\varepsilon}\varphi_3(t_{\tau_1},t)$. It is known that,

$$\frac{1}{2\pi i}\int_{D_k^{\cdot}}\frac{d\tau_k}{\tau_k - z_k} = \begin{cases} 1, & z_k \in D_k^{+} \\ \frac{1}{2}, & z_k \in D_k^{\cdot} \\ 0, & z_k \in D_k^{-} \end{cases} \qquad (7)$$

Taking into account (7), it follows that ε→0 $\overset{0}{S}_{\varepsilon} \to \frac{1}{8}$.

$\overset{3}{S}_{\varepsilon}\varphi_3(\tau,t)$; $\overset{2}{S}_{\varepsilon}\varphi_3(\tau_{t_k},t)$ $(k=1,2,3)$; $\overset{1}{S}_{\varepsilon}\varphi_3(t_{\tau_k},t)$ $(k=1,2,3)$ are ε→0 respectively

$\overset{3}{\psi}_{\square}(t)$, $\frac{1}{2}\overset{2}{\psi}_1(t)$, $\frac{1}{2}\overset{2}{\psi}_2(t)$, $\frac{1}{2}\overset{2}{\psi}_3(t)$, $\frac{1}{4}\overset{1}{\psi}_1(t)$, $\frac{1}{4}\overset{1}{\psi}_2(t)$, $\frac{1}{4}\overset{1}{\psi}_3(t)$,

Let's denote it as, in this example

$$\overset{3}{\psi}_{\square}(t_1,t_2,t_3) = \frac{1}{(2\pi i)^3}\int_{\Delta}\frac{\varphi_3(\tau,t)}{\prod_{k=1}^{3}(\tau_k - t_k)}d\tau \qquad (8)$$

$$\overset{2}{\psi}_{1\,\square}(t_1,t_2,t_3) = \frac{1}{(2\pi i)^2}\int_{D_2^{\cdot}}\int_{D_3^{\cdot}}\frac{\varphi_3(\tau_{t_1},t)}{(\tau_2 - t_2)(\tau_3 - t_3)}d\tau_2 d\tau_3 \qquad (9)$$

$$\overset{1}{\psi}_{1\,\square}(t_1,t_2,t_3) = \frac{1}{2\pi i}\int_{D_1^{\cdot}}\frac{\varphi_3(t_{\tau_1},t)}{\tau_1 - t_1}d\tau_1 \qquad (10).$$

Thus, the initial value of the integral (2), When $n = 3$:

$$\Phi(t_1,t_2,t_3) = \overset{3}{\psi}(t_1,t_2,t_3) + \frac{1}{2}\overset{2}{\psi}_1(t_1,t_2,t_3) + \frac{1}{2}\overset{2}{\psi}_2(t_1,t_2,t_3) +$$

$$+\frac{1}{2}\overset{2}{\psi}_3(t_1,t_2,t_3) + \frac{1}{4}\overset{1}{\psi}_1(t_1,t_2,t_3) + \frac{1}{4}\overset{1}{\psi}_2(t_1,t_2,t_3) + \frac{1}{4}\overset{1}{\psi}_3(t_1,t_2,t_3) +$$

$$+\frac{1}{8}\varphi(t_1,t_2,t_3) \qquad (11)$$

будет.

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