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Class of Functions Satisfying the Holder Condition

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Abstract: $\varphi(t_1,t_2,\dots,t_n)$ is about the satisfaction of the Golder condition in the multivariable complex plane.

Let the function φ(t) be defined in a Δ-domain.

Keywords: If $\varphi(t)=\varphi(t_1,t_2,\dots,t_n)$ is a function such that, for points in the domain $\varphi(t)=\varphi(t_1,t_2,\dots,t_n)$.

$$|\varphi(t)-\varphi(\tau)| \le \sum_{k=1}^{n} A_k\,|t_k-\tau_k|^{\alpha_k} \qquad (1)$$

then the function $\varphi(t)$ is said to satisfy the Holder condition in Δ, where $A_k \quad (k=\overline{1,n})$ are positive constants known as the Holder constants, and α_k, $0<\alpha_k\le 1 \ (k=\overline{1,n})$ - are the invariant exponents satisfying the inequality, which are called the Holder exponents

We define the class of functions that satisfy the Holder condition as $H_{\alpha_k}(\Delta)$.

If $\varphi(t)\in H_{\alpha_k}(\Delta)$, то $\varphi(t)\in H_{\beta_k}(\Delta) \ \ (\beta_k<\alpha_k,\ k=\overline{1,n})$.

Indeed, according to the condition $\varphi(t)\in H_{\alpha_k}(\Delta)$;

$$|\varphi(t)-\varphi(\tau)| \le \sum_{k=1}^{n} A_k\,|t_k-\tau_k|^{\alpha_k}$$

$$\sum_{k=1}^{n} A_k\,|t_k-\tau_k|^{\alpha_k} = \sum_{k=1}^{n} A_k\,|t_k-\tau_k|^{\alpha_k-\beta_k}\cdot|t_k-\tau_k|^{\beta_k} \le$$

$\le \sum_{k=1}^{n} \breve{A}_k\,|t_k-\tau_k|^{\beta_k} \quad \Rightarrow \ \varphi(t)\in H_{\beta_k}(\Delta)$ It follows from this that.

1-Theorem. The function φ(t) satisfies the Holder condition (1) in Δ other for each argument t_p для $t_k \ (p\ne k, k=\overline{1,n})$ with respect to the arguments is flat necessary and sufficient to satisfy the Holder condition.

Proof. Indeed, $\tau_k=t_k$, $k=\overline{1,n}\,, p\ne k$, when

$$\left|\varphi(t)-\varphi(t_{\tau_p})\right| \le A_p\left|t_p-\tau_p\right|^{\alpha_p}, \quad p=\overline{1,n} \qquad (2)$$

will be appropriate. Similarly, it is not difficult to prove the following inequalities:

$$\begin{cases} \left|\varphi\left(t_{\tau_p}\right)-\varphi(t_{\tau_{pq}})\right| \le A_q\left|t_q-\tau_q\right|^{\alpha_q}, \quad p,q=\overline{1,n}, p<q \\ \left|\varphi\left(t_{\tau_{pq}}\right)-\varphi(t_{\tau_{pqr}})\right| \le A_r|t_r-\tau_r|^{\alpha_r}, \quad p,q,r=\overline{1,n}, p<q<r \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \left|\varphi\left(\tau_{t_p}\right)-\varphi(\tau_{t_{pq}})\right| \le A_q\left|t_q-\tau_q\right|^{\alpha_q}, \quad p,q=\overline{1,n}, p<q \\ \left|\varphi(\tau)-\varphi(\tau_{t_p})\right| \le A_p\left|t_p-\tau_p\right|^{\alpha_p}, \quad p=\overline{1,n} \end{cases} \qquad (3)$$

(3) taking into account the system of inequalities

$$|\varphi(t)-\varphi(\tau)| \le \left|\varphi(t)-\varphi(t_{\tau_1})\right| + \left|\varphi(t_{\tau_1})-\varphi(t_{\tau_{12}})\right| + \dots +$$

$$+\left|\varphi(\tau_{t_{n-1,n}})-\varphi(\tau_{t_n})\right| + \left|\varphi(\tau_{t_n})-\varphi(\tau)\right|,$$

The inequality implies the statement of the theorem.

Some sums are used later.

$$\varphi_n(\tau;t)=\varphi_n(\tau_1,\tau_2,\dots,\tau_n;t_1,t_2,\dots,t_n)$$

we define the following sum through

$$\varphi_n(\tau;t)=\varphi(\tau)-\sum_{p=1}^{n}\varphi\left(\tau_{t_p}\right)+\sum_{p=1}^{n}\sum_{q=1}^{n}\varphi\left(\tau_{t_{pq}}\right)-\dots+$$

$$+(-1)^{n-2}\sum_{p=1}^{n}\sum_{q=1}^{n}\varphi\left(t_{\tau_{pq}}\right)+(-1)^{n-1}\sum_{p=1}^{n}\varphi\left(t_{\tau_p}\right)+(-1)^n\varphi(t), p<q \qquad (4)$$

from this $\varphi_n(\tau;t)=(-1)^n\varphi_n(t;\tau)$ (5)

It is easy to check the correctness of the equality.

For example, when n=3

$$\varphi_3(\tau;t)=\varphi(\tau_1,\tau_2,\tau_3)-\varphi(\tau_1,\tau_2,t_3)-\varphi(\tau_1,t_2,\tau_3)-\varphi(t_1,\tau_2,\tau_3)+$$

$$+\varphi(\tau_1,t_2,t_3)+\varphi(t_1,\tau_2,t_3)+\varphi(t_1,t_2,\tau_3)-\varphi(t_1,t_2,t_3) \qquad (6)$$

$$\varphi_n\left(\tau_{t_p};t\right)=\varphi_n(\tau_1,\tau_2,\dots,\tau_{p-1},t_p,\tau_{p+1},\dots,\tau_n;t_1,t_2,\dots,t_n)$$

through this

$$\varphi_n\left(\tau_{t_p};t\right)=\varphi\left(\tau_{t_p}\right)-\sum_{\substack{q=1\\q\ne p}}^{n}\varphi\left(\tau_{t_{pq}}\right)+\sum_{\substack{q=1\\q,r\ne p}}^{n}\sum_{\substack{r=1\\q<r}}^{n}\varphi\left(\tau_{t_{pqr}}\right)-\dots+$$

$$+(-1)^{n-3}\sum_{\substack{q=1\\ q,r\neq p}}^{n}\sum_{\substack{r=1\\ q<r}}^{n}\varphi\left(t_{\tau_{qr}}\right)+(-1)^{n-2}\sum_{\substack{q=1\\ q\neq p}}^{n}\varphi\left(t_{\tau_q}\right)+(-1)^{n+1}\varphi(t) \qquad 7)$$

Sum (7) has 2^{n-1}terms and is constructed similarly to (6), but with one less term.

When $n=3$, the representation of (7) looks like this:

$$\begin{cases}\varphi_3\left(\tau_{t_3};t\right)=\varphi_3\left(t_{\tau_{12}};t\right)=\varphi(\tau_1,\tau_2,t_3)-\varphi(\tau_1,t_2,t_3)-\varphi(t_1,\tau_2,\tau_3)+\varphi(\tau_1,\tau_2,t_3)\\ \varphi_3\left(\tau_{t_2};t\right)=\varphi_3\left(t_{\tau_{13}};t\right)=\varphi(\tau_1,t_2,\tau_3)-\varphi(\tau_1,t_2,t_3)-\varphi(t_1,t_2,\tau_3)+\varphi(t_1,t_2,t_3)\\ \varphi_3\left(\tau_{t_1};t\right)=\varphi_3\left(t_{\tau_{23}};t\right)=\varphi(t_1,\tau_2,\tau_3)-\varphi(t_1,t_2,\tau_3)-\varphi(t_1,\tau_2,t_3)+\varphi(t_1,t_2,t_3)\end{cases} \quad (8)$$

Similarly, $\varphi_n\left(\tau_{t_{pq}};t\right)$, $\varphi_n\left(\tau_{t_{pqr}};t\right),\ldots,\varphi_n\left(t_{\tau_{pq}};\,t,t\right)$and $\varphi_n\left(t_{\tau_p};t,\right)$ etc., are also constructed for sums. when n=3, the visibility of these sums will be:

$$\begin{cases}\varphi_3\left(\tau_{t_{12}};t\right)=\varphi_3\left(t_{\tau_3};t\right)=\varphi(t_1,t_2,\tau_3)-\varphi(t_1,t_2,t_3)\\ \varphi_3\left(\tau_{t_{13}};t\right)=\varphi_3\left(t_{\tau_2};t\right)=\varphi(t_1,\tau_2,t_3)-\varphi(t_1,t_2,t_3)\\ \varphi_3\left(\tau_{t_{23}};t\right)=\varphi_3\left(t_{\tau_1};t\right)=\varphi(\tau_1,t_2,t_3)-\varphi(t_1,t_2,t_3)\end{cases} \qquad (9)$$

(6), as a result of adding the two above mentioned systems and $\varphi(t_1,t_2,t_3)$ this

$$\varphi(\tau_1,\tau_2,\tau_3)=\varphi_3(\tau;t)+\varphi_3\left(\tau_{t_1};t\right)+\varphi_3\left(\tau_{t_2};t\right)+\varphi_3\left(\tau_{t_3};t\right)++\varphi_3\left(t_{\tau_1};t\right)+$$

$$+\varphi_3\left(t_{\tau_2};t\right)+\varphi_3\left(t_{\tau_3};t\right)+\varphi(t_1,t_2,t_3) \qquad (10)$$

we form a cycle. In general, the appearance of this mirror will be as follows.

$$\varphi(\tau)=\varphi_n(\tau;t)+\sum_{p=1}^{n}\varphi_n(\tau_{t_p};t)+\sum_{\substack{p=1\\ p<q}}^{n}\sum_{q=1}^{n}\varphi_n\left(\tau_{t_{pq}};t\right)+\cdots+$$

$$+\sum_{\substack{p=1\\ p<q}}^{n}\sum_{q=1}^{n}\varphi_n\left(t_{\tau_{pq}};t\right)+\sum_{p=1}^{n}\varphi_n(t_{\tau_p};t)+\varphi(t) \qquad (11)$$

Checking the correctness of this statement is not as difficult as stated above.

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Let us divide the members of the sum (4) into two groups such that each group is described as a subtraction of functions that differ only in one specific argument. In the sum (4), the number of such groups is 2^{n-1}. On the other hand, the number of groupings will be exactly equal to n.

For example, for the sum (4), When $n=3$:

$$\begin{cases} \varphi_3(\tau;t) = [\varphi(\tau_1,\tau_2,\tau_3) - \varphi(\tau_1,\tau_2,t_3)] + [\varphi(\tau_1,t_2,t_3) - \varphi(\tau_1,t_2,\tau_3)] \\ \quad +[\varphi(t_1,\tau_2,t_3) - \varphi(t_1,\tau_2,\tau_3)] + [\varphi(t_1,t_2,\tau_3) - \varphi(t_1,t_2,t_3)], \\ \varphi_3(\tau;t) = [\varphi(\tau_1,\tau_2,\tau_3) - \varphi(\tau_1,t_2,\tau_3)] + [\varphi(\tau_1,t_2,t_3) - \varphi(\tau_1,\tau_2,t_3)] \\ \quad +[\varphi(t_1,\tau_2,t_3) - \varphi(t_1,t_2,t_3)] + [\varphi(t_1,t_2,\tau_3) - \varphi(t_1,\tau_2,\tau_3)], \\ \varphi_3(\tau;t) = [\varphi(\tau_1,\tau_2,\tau_3) - \varphi(t_1,\tau_2,\tau_3)] + [\varphi(\tau_1,t_2,t_3) - \varphi(t_1,t_2,t_3)] \\ \quad +[\varphi(t_1,\tau_2,t_3) - \varphi(\tau_1,\tau_2,t_3)] + [\varphi(t_1,t_2,\tau_3) - \varphi(\tau_1,t_2,\tau_3)]. \end{cases}$$

Similar considerations can be applied to the sums

$$\varphi_n\left(\tau_{t_p};t\right),\ \varphi_n\left(\tau_{t_{pq}};t\right),\dots,\varphi_n\left(t_{\tau_{pq}};t\right) \text{и}\ \varphi_n\left(t_{\tau_p};t\right).$$

In this section we evaluate the sums considered in (4); to do this we first look at n=3. From the above system we find from (4) that the modulus of the sum is less than or equal to the sum of the moduli:

$$|\varphi_3(\tau;t)| \le 4A_k|\tau_k - t_k|^{\alpha_k}, \quad k = 1,2,3.$$

With bosonism it is clear that in the general case:

$$\begin{cases} |\varphi_n(\tau;t)| \le 2^{n-1}A_k|\tau_k - t_k|^{\alpha_k}, \quad k = \overline{1,n} \\ \left|\varphi_n\left(\tau_{t_p};t\right)\right| \le 2^{n-2}A_k|\tau_k - t_k|^{\alpha_k}, \quad k \ne p;\ k,p = \overline{1,n} \\ \left|\varphi_n\left(\tau_{t_{pq}};t\right)\right| \le 2^{n-3}A_k|\tau_k - t_k|^{\alpha_k}, \quad k \ne p,q;\ p<q;\ \ k,p,q = \overline{1,n} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \\ \left|\varphi_n\left(t_{\tau_{pqr}};t\right)\right| \le 2^2A_k|\tau_k - t_k|^{\alpha_k}, \quad p<q<r;\ k\ne p,q,r;\ \ k,p,q,r = \overline{1,n} \\ \left|\varphi_n\left(t_{\tau_{pq}};t\right)\right| \le 2A_k|\tau_k - t_k|^{\alpha_k}, \quad k \ne p,q;\ p<q;\ \ k,p,q = \overline{1,n} \end{cases} \tag{12}$$

this is what we find using the first one:

$$|\varphi_n(\tau;t)| \le [|\varphi_n(\tau;t)|^n]^{\frac{1}{n}} \le \left[\prod_{k=1}^{n} 2^{n-1}A_k\,|\tau_k - t_k|^{\alpha_k}\right]^{\frac{1}{n}} =$$

$$= 2^{n-1}\prod_{k=1}^{n} A_k^{\frac{1}{n}}|\tau_k - t_k|^{\frac{\alpha_k}{n}}$$

After similar reasoning, we can obtain the following inequalities for the remaining inequalities of the above system:

$$\left|\varphi_n\left(t_{\tau_{pq}};t\right)\right| \le 2\sqrt{A_p\cdot A_q}\,|\tau_p - t_p|^{\frac{\alpha_p}{2}}\cdot|\tau_q - t_q|^{\frac{\alpha_q}{2}};$$

$$p,q = \overline{1,n};\ \ p<q, \tag{13}$$

$$\left|\varphi_n\left(t_{\tau_{pqr}};t\right)\right| \le 4\sqrt[3]{A_p\cdot A_q\cdot A_r}\,|\tau_p - t_p|^{\frac{\alpha_p}{3}}\cdot|\tau_q - t_q|^{\frac{\alpha_q}{3}}\cdot|\tau_r - t_r|^{\frac{\alpha_r}{3}};$$

$$p<q<r;\ \ p,q,r = \overline{1,n}; \tag{14}$$

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$$\left|\varphi_n\left(\tau_{t_{pq}};t\right)\right| \leq 2^{n-3}\prod_{k=1}^{n} A_k^{\frac{1}{n-2}}\left|\tau_p - t_p\right|^{\frac{\alpha_k}{n-2}};$$

$$p,q=\overline{1,n};\quad p<q\quad k\neq p,q \qquad (15)$$

$$\left|\varphi_n\left(\tau_{t_p};t\right)\right| \leq 2^{n-2}\prod_{k=1}^{n} A_k^{\frac{1}{n-1}}\left|\tau_k - t_k\right|^{\frac{\alpha_k}{n-1}};\quad p=\overline{1,n};\quad k\neq p, \qquad (16)$$

$$\left|\varphi_n(\tau;t)\right| \leq 2^{n-1}\prod_{k=1}^{n} A_k^{\frac{1}{n}}\left|\tau_k - t_k\right|^{\frac{\alpha_k}{n}}; \qquad (17)$$

We write the first of (3) through the sum

$$\left|\varphi_n\left(t_{\tau_p},t\right)\right| \leq A_p\left|\tau_p - t_p\right|^{\alpha_p},\quad p=\overline{1,n} \qquad (18)$$

note that for cases n=2,3 we obtain estimates for the same sums:

$$\left|\varphi_2\left(t_{\tau_p},t\right)\right| \leq A_p\left|\tau_p - t_p\right|^{\alpha_p},\quad p=\overline{1,2} \qquad (19)$$

$$\left|\varphi_2(\tau;t)\right| \leq 2\sqrt{A_1\cdot A_2}\,\left|\tau_1 - t_1\right|^{\frac{\alpha_1}{2}}\cdot\left|\tau_2 - t_2\right|^{\frac{\alpha_2}{2}}, \qquad (20)$$

$$\left|\varphi_3\left(t_{\tau_p},t\right)\right| \leq A_p\left|\tau_p - t_p\right|^{\alpha_p},\quad p=\overline{1,3} \qquad (21)$$

$$\left|\varphi_3\left(t_{\tau_p};t\right)\right| \leq 2\prod_{k=1}^{3} A_k^{\frac{1}{2}}\left|\tau_k - t_k\right|^{\frac{\alpha_k}{2}};\quad p=\overline{1,3};\quad k\neq p, \qquad (22)$$

$$\left|\varphi_3(\tau;t)\right| \leq 4\prod_{k=1}^{3} A_k^{\frac{1}{3}}\left|\tau_k - t_k\right|^{\frac{\alpha_k}{3}}; \qquad (23)$$

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