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On a Cubic Stochastic Operator

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Abstract: In this paper, we consider a class of cubic stochastic operators defined on a finite-dimensional simplex. Namely, cubic stochastic operators that have the form of a product of three linear operators defined on a simplex.

Introduction

There are many systems which are described by nonlinear operators. A cubic stochastic operator (CSO) is one of the simplest nonlinear cases.

Let $E=\{1,2,...,m\}$ be a finite set and the set of all probability distribution on E

$S^{m-1}=\{x=(x_1,\dots,x_m)\in\mathbb{R}^m: x_i\geq 0, \sum_{i=1}^m x_i=1\}$ (1)

be the $(m-1)$-dimensional simplex. A CSO is a mapping defined as $W:S^{m-1}\to S^{m-1}$ of the simplex into itself, of the form $W(\mathbf{x})=\mathbf{x}'\in S^{m-1}$, where

$$x_l'=\sum_{i,j,k=1}^m P_{ijk,l}x_ix_jx_k, \quad l\in E, \quad (2)$$

and the coefficients $P_{ijk,l}$ satisfy

$$P_{ijk,l}\geq 0, \quad \sum_{l=1}^m P_{ijk,l}=1 \text{ for all } i,j,k\in E. \quad (3)$$

The trajectory (orbit) $\{\mathbf{x}^{(n)}\}_{n\geq 0}$, of W for an initial value $\mathbf{x}^{(0)}\in S^{m-1}$ is defined by

$$\mathbf{x}^{(n+1)}=W\left(\mathbf{x}^{(n)}\right)=W^{(n+1)}\left(\mathbf{x}^{(0)}\right), \qquad n=0,1,2,\dots .$$

One of the main problems in mathematical biology is to study the asymptotic behavior of the trajectories.

For a given $\mathbf{x}^{(0)}\in S^{m-1}$, the trajectory $\{\mathbf{x}^{(n)}\}_{n\geq 0}$ of initial point $\mathbf{x}^{(0)}$ under action of CSO is defined by $\mathbf{x}^{(n+1)}=W\left(\mathbf{x}^{(n)}\right)$, where $n=0,1,2,\dots$ with $\mathbf{x}=\mathbf{x}^{(0)}$. Denote by $\omega\left(\mathbf{x}^{(0)}\right)$ the set of limit points of the trajectory $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}$. Since $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}\subset S^{m-1}$ and S^{m-1} is a compact set, it is follows that $\omega\left(\mathbf{x}^{(0)}\right) \neq \varnothing$. If $\omega\left(\mathbf{x}^{(0)}\right)$ consists of a single point, they the trajectory converges and $\omega\left(\mathbf{x}^{(0)}\right)$ is a fixed point of the operator W. A point $\mathbf{x} \in S^{m-1}$ is called a fixed of the W if $W(\mathbf{x}) = \mathbf{x}$. Denote by $\mathrm{Fix}(W)$ the set of all fixed points of the operator W, i.e.

$$\mathrm{Fix}(W) = \left\{\mathbf{x} \in S^{m-1} : W(\mathbf{x}) = \mathbf{x}\right\}.$$

Let $DW\left(\mathbf{x}^*\right) = \left(\partial W_i / \partial x_j\right)\left(\mathbf{x}^*\right)$ be a Jacobian of W at the point $\mathbf{x}^*$.

The CSO $W : S^2 \to S^2$ is:

$$W : \begin{cases} x_1' = x_1\left(x_1 + (1-\alpha)x_2 + x_3\right)\left(x_1 + x_2 + \left(1 - \dfrac{\alpha}{2}\right)x_3\right), \\ x_2' = x_2\left(x_1 + x_2 + (1-\alpha)x_3\right)\left((1+\alpha)x_1 + x_2 + x_3\right), \\ x_3' = x_3\left(\left(1 + \dfrac{\alpha}{2}\right)x_1 + x_2 + x_3\right)\left(x_1 + (1+\alpha)x_2 + x_3\right), \end{cases} \quad (3)$$

where $\alpha \in [0,1]$.

Using the equation $x_1 + x_2 + x_3 = 1$ we rewrite the operator (3) as follows

$$W : \begin{cases} x_1' = x_1\left(1 - \alpha x_2\right)\left(1 - \dfrac{\alpha}{2}x_3\right), \\ x_2' = x_2\left(1 - \alpha x_3\right)\left(1 + \alpha x_1\right), \\ x_3' = x_3\left(1 + \dfrac{\alpha}{2}x_1\right)\left(1 + \alpha x_2\right). \end{cases} \quad (4)$$

Evidently, that if $\alpha = 0$ the CSO (4) is the identity map. For this in the below, we consider the case when $\alpha \neq 0$.

Let a face of the simplex S^2 be the set $\Gamma_\gamma = \left\{\mathbf{x} \in S^2 : x_i = 0,\ i \notin \gamma \subset \{1,2,3\}\right\}$.

Let the set $\mathrm{int}\, S^2 = \left\{\mathbf{x} \in S^2 : x_1 x_2 x_3 > 0\right\}$ and let the set $\partial S^2 = S^2 \setminus \mathrm{int}\, S^2$ be the interior and the boundary of the simplex S^2, respectively. Let $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, $\mathbf{e}_3 = (0,0,1)$ be the vertexes of the two-dimensional simplex.

Let $\mathbf{x}^{(0)} \in S^2$ be the initial point. Then the trajectory of $\mathbf{x}^{(0)}$ is denote by $\left\{\mathbf{x}^{(n)}\right\}_{n \geq 0}$ and is defined as $W\left(\mathbf{x}^{(n)}\right) = \mathbf{x}^{(n+1)}$ where $n = 0, 1, 2, \ldots$.

Theorem 1. *For the CSO* W (4), *the following assertions true*:

(i) *If* $a \in [-1,0)$, *then*

$$\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \begin{cases} \mathbf{e}_2 & \text{if} \quad \mathbf{x}^{(0)} \in \Gamma_{\{2,3\}} \setminus \{\mathbf{e}_3\}, \\ \mathbf{e}_1 & \text{if} \quad \mathbf{x}^{(0)} \in S^2 \setminus \Gamma_{\{2,3\}}. \end{cases}$$

(ii) *If* $a \in (0,1]$, *then*

$$\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \begin{cases} \mathbf{e}_2 & \text{if} \quad \mathbf{x}^{(0)} \in \Gamma_{\{1,2\}} \setminus \{\mathbf{e}_1\}, \\ \mathbf{e}_3 & \text{if} \quad \mathbf{x}^{(0)} \in S^2 \setminus \Gamma_{\{1,2\}}. \end{cases}$$

Proof: (i) If $a \in [-1,0)$. Let $\mathbf{x}^{(0)} = \left(0, x_2^{(0)}, x_3^{(0)}\right) \in \Gamma_{\{2,3\}}$. By Theorem 1, the face $\Gamma_{\{2,3\}}$ is an invariant set and the vertexes $\mathbf{e}_2$ and $\mathbf{e}_3$ are fixed points belonging to this face. For the SCSO W (11) is true $x_1^{(n)} = 0$, $n = 0,1,2,\ldots$ at the face $\Gamma_{\{2,3\}}$. The restriction of the SCSO W (11) on the face $\Gamma_{\{2,3\}}$ has the form

$$\begin{cases} x_2' = x_2\left(x_2 + (1-\alpha)x_3\right), \\ x_3' = x_3\left((1+\alpha)x_2 + x_3\right). \end{cases} \quad (5)$$

From the first equation of (5), one has

$$x_2^{(n+1)} = x_2^{(n)}\left(1 - \alpha x_3^{(n)}\right) \geq x_2^{(n)}, \quad n = 0,1,2,\ldots.$$

Therefore, it follows that there exists the $\lim_{n\to\infty} x_2^{(n)} = x_2^*$. Moreover, from the second equation of (5), one has

$$x_3^{(n+1)} = x_3^{(n)}\left(1 + \alpha x_2^{(n)}\right) \leq x_3^{(n)}, \quad n = 0,1,2,\ldots.$$

Therefore, it follows that there exists the $\lim_{n\to\infty} x_3^{(n)} = x_3^*$. So there is the limit $\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \mathbf{x}^* = \left(0, x_2^*, x_3^*\right)$. Since $\mathbf{x}^*$ should be a fixed we get $\mathbf{x}^* = \mathbf{e}_2$. That is, if $x_2^{(0)} > 0$ then we have that $\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \mathbf{e}_2$ for any $\mathbf{x}^{(0)} = \Gamma_{\{2,3\}} \setminus \{\mathbf{e}_3\}$.

Let $\mathbf{x}^{(0)} = S^2 \setminus \Gamma_{\{2,3\}}$. Then, from the first equation of operator (5), one has

$$x_1^{(n+1)} = x_1^{(n)}\left(1 - \frac{\alpha}{2} x_3^{(n)}\right)\left(1 - \alpha x_2^{(n)}\right) \geq x_1^{(n)}, \quad n = 0,1,2,\ldots.$$

So we have that there exists the $\lim_{n\to\infty} x_1^{(n)} = x_1^*$. Moreover, from the third equation of operator (5), one has

$$x_3^{(n+1)} = x_3^{(n)}\left(1+\frac{\alpha}{2}x_1^{(n)}\right)\left(1+\alpha x_2^{(n)}\right) \le x_3^{(n)}, \quad n = 0,1,2,\dots .$$

Therefore, it follows that there exists the $\lim_{n\to\infty} x_3^{(n)} = x_3^*$. Thus, using $x_1^{(n)} + x_2^{(n)} + x_3^{(n)} = 1$ we have $\lim_{n\to\infty} x_2^{(n)} = x_2^*$. So there is the limit $\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \mathbf{x}^* = \left(x_1^*, x_2^*, x_3^*\right)$. Since $\mathbf{x}^*$ should be a fixed point we get $\mathbf{x}^* = \mathbf{e}_1$.

Consequently, we have that $\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \mathbf{e}_1$ for any $\mathbf{x}^{(0)} = S^2 \setminus \Gamma_{\{2,3\}}$.

(ii) If $\alpha \in (0,1]$. Let $\mathbf{x}^{(0)} \in \Gamma_{\{1,2\}}$. By Theorem 1, the face $\Gamma_{\{1,2\}}$ is an invariant set and the vertexes $\mathbf{e}_1$ and $\mathbf{e}_2$ are fixed points belonging to this face. The restriction of the SCSO W on the face $\Gamma_{\{1,2\}}$ has the form

$$\begin{cases} x_1' = x_1\left(x_1 + (1-\alpha)x_2\right), \\ x_2' = x_2\left((1+\alpha)x_1 + x_2\right). \end{cases} \quad (6)$$

Not that, for the CSO W is true $x_3^{(n)} = 0, \ n = 0,1,2,\dots$ at the face $\Gamma_{\{1,2\}}$. From the first equation of (6), one has

$$x_1^{(n+1)} = x_1^{(n)}\left(1-\alpha x_2^{(n)}\right) \le x_1^{(n)}, \quad n = 0,1,2,\dots .$$

Therefore, it follows that there exists the $\lim_{n\to\infty} x_1^{(n)} = x_1^*$. Moreover, from the second equation of (6), one has

$$x_2^{(n+1)} = x_2^{(n)}\left(1+\alpha x_1^{(n)}\right) \ge x_2^{(n)}, \quad n = 0,1,2,\dots .$$

So we have that there exists the $\lim_{n\to\infty} x_2^{(n)} = x_2^*$. Hence there is the limit $\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \mathbf{x}^* = \left(x_1^*, x_2^*, 0\right)$. Since $\mathbf{x}^*$ should be a fixed point we get $\mathbf{x}^* = \mathbf{e}_2$. That is, if $x_2^{(0)} > 0$ then we have that $\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \mathbf{e}_2$ for any $\mathbf{x}^{(0)} = \Gamma_{\{1,2\}} \setminus \left\{\mathbf{e}_1\right\}$.

Let $\mathbf{x}^{(0)} = S^2 \setminus \Gamma_{\{1,2\}}$. Then, from the first equation of operator (6), one has

$$x_1^{(n+1)} = x_1^{(n)}\left(1-\frac{\alpha}{2}x_3^{(n)}\right)\left(1-\alpha x_2^{(n)}\right) \le x_1^{(n)}, \quad n = 0,1,2,\dots .$$

Therefore, it follows that there exists the $\lim_{n\to\infty} x_1^{(n)} = x_1^*$. Moreover, from the third equation of operator (6), one has

$$x_3^{(n+1)} = x_3^{(n)}\left(1+\frac{\alpha}{2}x_1^{(n)}\right)\left(1+\alpha x_2^{(n)}\right) \geq x_3^{(n)}, \quad n=0,1,2,\dots .$$

Hence one has that there exists the $\lim_{n\to\infty} x_3^{(n)} = x_3^*$. Thus, using $x_1^{(n)} + x_2^{(n)} + x_3^{(n)} = 1$ we have $\lim_{n\to\infty} x_2^{(n)} = x_2^*$. So there is the limit $\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \mathbf{x}^* = \left(x_1^*, x_2^*, x_3^*\right)$. Since $\mathbf{x}^*$ should be a fixed point we get $\mathbf{x}^* = \mathbf{e}_3$.

Consequently, we have that $\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \mathbf{e}_3$ for any $\mathbf{x}^{(0)} = S^2 \setminus \Gamma_{\{1,2\}}$.

The theorem is proved. □

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