Quest: Journal of Geometry, Mathematical and Quantum Physics (Quest)

e-ISSN: xxxx-xxxx Vol.1 Issue 2|2024

Numerical Comparison for Solution of Fredholm Integro-Differential Equations

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Abstract: This study explores four prominent methods for solving integral and integro-differential equations: the Homotopy Perturbation Method, the Modified Adomian Decomposition Method, the Variational Iteration Method, and the Adomian Decomposition Method. While each approach has been widely used, a clear understanding of their comparative strengths remains underexplored, particularly for problems with known exact solutions. To address this gap, the study applies these methods to Fredholm's integro-differential equations and evaluates their accuracy. Results show that all methods yield successful approximate solutions, with differences in performance highlighted. These findings provide valuable insights into method selection for solving similar equations, advancing numerical analysis in applied mathematics.

Keywords: Adomian, Variational, Homotopy, Fredholm, Integral Equation

1. Introduction

Numerous scientific disciplines use the Fredholm integral and integro-differential equations. Furthermore, the derivation of Fredholm integral equations from boundary value issues was observed. Most people are familiar with Erik Ivar Fredholm (1866–1927) from his work on spectrum theory and integral equations. Assume that the standard form Fredholm integro-differential equation .

$$\sum_{j=0}^{k} p_j(x)u^{(j)}(x) = f(x) + \lambda \int_a^b K(x,t)G(u(t))dt \qquad (1)$$

with the initial conditions

$$u^{(r)}(a) = b_r,\ r = 0,1,2,\cdots,(k-1) \tag{2}$$

where $u^{(j)}(x)$ is the j^{th} derivative of the unknown function $u(x)$ that will be determined, $K(x,t)$ is the kernel of the equation, $f(x)$ and $p_j(x)$ are an analytic function, G is non-linear function of u and a, b, λ, and b_r are real finite constants. Studying the behavior of the solution that can be explicitly ascertained by semi-analytical approximate methods such as ADM, MADM, VIM, and HPM is the primary objective of this chapter. We established the solutions to the Fredholm integro-differential equations' existence and uniqueness .

2. Materials and Methods

Prominent methods such as the Adomian Decomposition Method [1-4], Modified Adomian Decomposition Method [3,5,6], Variational Iteration Method [7,8,9], and Homotopy Perturbation Method [10,11,12] have significantly advanced the development of sophisticated and practical techniques for solving Fredholm integro-differential equations. This section will thoroughly review each of these techniques

2.1 Adomian Decomposition Method(ADM)

Now, we rewrite eq (1) in the form

$$p_k(x)u^k(x) + \sum_{j=0}^{k-1} p_j(x)u^j(x) = f(x) + \lambda \int_a^b K(x,t)G(u(t))dt \tag{3}$$

Then

$$u^k(x) = \frac{f(x)}{p_k(x)} + \lambda \int_a^b \frac{K(x,t)}{p_k(x)} G(u(t))dt - \sum_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)} u^j(x)$$

To obtain the approximate solution, we integrating (k)-times in the interval [a,x] with respect to x we get.

$$u(x) = L^{-1}\left(\frac{f(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r + \lambda L^{-1}\left(\int_a^b \frac{K(x,t)}{p_k(x)} G(u(t))dt\right) \tag{4}$$

where L^{-1} is the multiple integration operator given as the following form

$$L^{-1}(\cdot) = \int_a^b \int_a^b \cdots \int_a^b (\cdot)dxdx\cdots dx\ (k-\text{ times})$$

Now we apply ADM

$$G(u(x)) = \sum_{n=0}^{\infty} A_n \tag{5}$$

where $A_n, n \geq 0$ are the Adomian polynomials determined formally as the following

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\mu^n} G\left(\sum_{i=0}^{\infty} \mu^i u_i\right)\right]\Bigg|_{\mu=0} \tag{6}$$

or equivalently

$$\begin{aligned} A_0 &= G(u_0) \\ A_1 &= u_1 G'(u_0) \\ A_2 &= u_2 G'(u_0) + \frac{1}{2!} u_1^2 G''(u_0) \\ A_3 &= u_3 G'(u_0) + u_1 u_2 G''(u_0) + \frac{1}{3!} u_1^3 G'''(u_0) \end{aligned}$$

The solution of u is represented by the following series in the conventional form of the decomposition approach .

$$u = \sum_{i=0}^{\infty} u_i \tag{7}$$

By substituting (5) and (7) in (4) we have

$$\begin{aligned} \sum_{i=0}^{\infty} u_i(x) = & L^{-1}\left(\frac{f(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r + \lambda \sum_{i=0}^{\infty} L^{-1}\left(\int_a^b \frac{K(x,t)}{p_k(x)} A_i(t)dt\right) \\ & - \sum_{i=0}^{\infty} \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_i^{(j)}(x)\right) \end{aligned}$$

The components $u_0, u_1, u_2, \cdots$ are usually determined recursively by

$$\begin{aligned} u_0 &= L^{-1}\left(\frac{f(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r \\ u_1 &= \lambda L^{-1}\left(\int_a^b \frac{K(x,t)}{p_k(x)} A_0(t)dt\right) - \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_0^{(j)}(x)\right) \end{aligned}$$

Then,

$$u(x) = \sum_{i=0}^{n} u_i$$

as the approximate solution.

2.2 Modified Adomian Decomposition Method (MADM)

Wazwaz highlighted the improved decomposition technique [6]. This approach

is predicated on the idea that the function f(x) is bifurcable into $f_1(x)$and $f_2(x)$ Based on this presumption, we established .

$$f(x) = f_1(x) + f_2(x) \tag{8}$$

When a function f has multiple components and can be divided into two distinct portions, we use this decomposition method. In this case, f is typically the product of a polynomial and a transcendental, trigonometric, or other function. It is crucial to select a convenient option for the portion $f_1(x)$. In order to increase the effectiveness of the procedure, we select $f_1(x)$to be one term of f, or if possible, a number of terms, and $f_2(x)$to be the other terms of f. From (8), we may write (3) in the form by utilizing the MADM.

$$p_k(x)u^k(x) + \sum_{j=0}^{k-1} p_j(x)u^j(x) = f_1(x) + f_2(x) + \lambda \int_a^b K(x,t)G(u(t))dt$$

Then

$$u^k(x) = \frac{f_1(x)}{p_k(x)} + \frac{f_2(x)}{p_k(x)} + \lambda \int_a^b \frac{K(x,t)}{p_k(x)} G(u(t))dt - \sum_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)} u^j(x)$$

To obtain the approximate solution, integrating (k)-times in the interval $[a,x]$ with respect to x, we get .

$$\begin{aligned} u(x) =& L^{-1}\left(\frac{f_1(x)}{p_k(x)}\right) + L^{-1}\left(\frac{f_2(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r \\ &+\lambda L^{-1}\left(\int_a^b \frac{K(x,t)}{p_k(x)} G(u(t))dt\right) - \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_n^{(j)}(x)\right) \end{aligned}$$

The components $u_0, u_1, u_2, \cdots$ are usually determined recursively by

$$u_0 = L^{-1}\left(\frac{f_1(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r$$

$$u_1 = L^{-1}\left(\frac{f_2(x)}{p_k(x)}\right) + \lambda L^{-1}\left(\int_a^b \frac{K(x,t)}{p_k(x)} A_0(t)dt\right) - \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_0^{(j)}(x)\right)$$

Then,

$$u(x) = \sum_{i=0}^{n} u_i$$

as the approximate solution.

2.3 Variational Iteration Method (VIM)

A wide class of non-linear and linear problems with approximations that

quickly converge to exact solutions can be solved using this technique. The idea behind this approach is to use universal Lagrange multipliers to build a corrective functional form. These multipliers ought to be selected so that the corrected value outperforms the trial function, the first approximation. In terms of trial function flexibility, it is the best. Thus, variational theory can be used to identify Lagrange multipliers [7,8,9]. You may get a thorough analysis of the variational iteration method in [5,10]. By setting boundary/beginning constraints, it becomes easy to choose the initial approximation with potential unknowns. To be precise, we take into consideration the general differential equation .

$$Lu(t) + Nu(t) = f(t)$$

where f(t) is a non-homogeneous term, N is a non-linear operator, and L is a linear operator. The terms of a sequence u_n are arranged in accordance with the variational iteration method [8] such that the sequence converges to the precise answer. A correction functional calculates the terms $u_n$as follows :

$$u_{n+1}(t) = u_n(t) + \int_0^t \xi(\tau)(Lu_n(\tau) + N\tilde{y}(\tau) - f(\tau))d\tau \quad (9)$$

The successive approximation u_n (t),n>0 of the solution u(t) will be easily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The zero approximation u_0 may be using any function that just satisfies at least the initial and boundary conditions. With ? determined, many approximations of u_n (t),n>0 follow directly.It has been observed that the VIM can solve a wide class of non-linear problems with approximations that rapidly converge to accurate solutions in an efficient, robust, and precise manner.To get the approximation solution of IVP (1)-(2), according to the VIM, the iteration formula (9) can be written as the following.

$$u_{n+1}(x) = u_n(x) + L^{-1}\left[\xi(x)\left[\sum_{j=0}^{k} p_j(x)u_n^{(j)}(x) - f(x) - \lambda\int_a^b K(x,t)G\big(u_n(t)\big)dt\right]\right] (10)$$

where L^{-1} is the multiple integration operator given as follows

$$L^{-1}(\cdot) = \int_a^b \int_a^b \cdots \int_a^b (\cdot)dxdx\cdots dx \ (k - \text{ times })$$

To find the optimal $\xi(x)$, we progress as the following

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta L^{-1}\left[\xi(x)\left[\sum_{j=0}^{k} p_j(x)u_n^{(j)}(x) - f(x) - \lambda\int_a^b K(x,t)G(u_n(t))dt\right]\right]$$

From (10), the stationary conditions can be obtained as follows

$$\xi'(x) = 0, \text{and } 1 + \xi(x)|_{x=t} = 0$$

In a solution, the Lagrange multipliers can be identified as $\xi(x) = -1$ and by substituting in (10), the following iteration formula is obtained .

$$u_0(x) = L^{-1}\left[\frac{f(x)}{p_k(x)}\right] + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!} b_r$$
$$u_{n+1}(x) = u_n(x) - L^{-1}\left[\sum_{j=0}^{k} p_j(x) u_n^{(j)}(x) - f(x) - \lambda \int_a^b K(x,t) G\big(u_n(t)\big) dt\right], n \geq 0 \quad (11)$$

The term $\sum_{r=0}^{k-1} \frac{(x-a)^r}{r!} b_r$ is obtained from the initial conditions, $p_k(x) \neq 0$. Relation (11) will enable us to determine the components $u_n(x)$ recursively for $n \geq 0$. Therefore, the approximation solution may be obtained by using .

$$u(t) = \lim_{n \to \infty} u_n(t)$$

2.4 Homotopy Perturbation Method (HPM)

He was the first to propose the homotopy perturbation approach [11,12,13, 14]. We examine the following non-linear differential equation to clarify the fundamental concept of this approach .

$$A(u) - f(r) = 0, \; r \in \Omega \quad (12)$$

under the boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \; r \in \Gamma \quad (13)$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω.

Overall, the operator A can be break up into two parts L and N, where L is linear, while N is non-linear eq (12). Hence, it can be rewritten as the following .

$$L(u) + N(u) - f(r) = 0 \quad (14)$$

By the homotopy technique [ ] [83]. We construct a homotopy $v(r,p): \Omega \times [0,1] \longrightarrow \mathbb{R}$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, p \in [0,1] \quad (15)$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0)] + p[N(v) - f(r)] = 0 \quad (16)$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of (12) which satisfies the boundary conditions. From (15), (16) we have .

$$H(v,0) = L(v) - L(u_0) = 0 \quad (17)$$
$$H(v,1) = A(v) - f(r) = 0 \quad (18)$$

The change in the process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and $L(v)-L(u_0)$ and $A(v)-f(r)$ are called homotopic. Now, suppose that the solution of (15), (16) can be expressed as .

$$v = v_0 + pv_1 + p^2v_2 + \cdots \tag{19}$$

The approximate solution of (12) can be obtained by putting $p = 1$.

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots \tag{20}$$

Then equating the terms with identical power of P, we obtain the following series of linear equations .

$$P^0: u_0(x) = \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r$$

$$P^1: u_1(x) = L^{-1}\left(\frac{f(x)}{p_k(x)}\right) + \lambda L^{-1}\left(\int_a^b \frac{K(x,t)}{p_k(x)} G(u_0(t))(t)dt\right) - \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_0^{(j)}(x)\right)$$

$$P^2: u_2(x) = \lambda L^{-1}\left(\int_a^b \frac{K(x,t)}{p_k(x)} G(u_1(t))(t)dt\right) - \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_1^{(j)}(x)\right)$$

$$P^3: u_3(x) = \lambda L^{-1}\left(\int_a^b \frac{K(x,t)}{p_k(x)} G(u_2(t))dt\right) - \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_2^{(j)}(x)\right)$$

3. Results and Discussion

Existence and Uniqueness Results

This section will present and demonstrate the existence and uniqueness of equation (1) with initial condition (2).
We write equation (1) in the form

$$u(x) = L^{-1}\left[\frac{f(x)}{p_k(x)}\right] + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!} b_r + \lambda_1 L^{-1}\left[\int_a^b \frac{1}{p_k(x)} K(x,t) G(u_n(t))dt\right] - L^{-1}\left[\sum_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)} u^{(j)}(x)\right]$$

Such that,

$$L^{-1}\left[\int_a^b \frac{1}{p_k(x)} K(x,t) G(u_n(t))dt\right] = \int_a^b \frac{(x-t)^k}{k!\, p_k(x)} K(x,t) G(u_n(t))dt$$

$$= \sum_{j=0}^{k-1} L^{-1}\left[\frac{p_j(x)}{p_k(x)}\right] u^{(j)}(x) = \sum_{j=0}^{k-1} \int_a^b \frac{(x-t)^{k-1} p_j(t)}{k-1!\, p_k(t)} u^{(j)}(t)dt$$

We set,

$$\Psi(x) = L^{-1}\left[\frac{f(x)}{p_k(x)}\right] + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!} b_r$$

Before beginning and proving the main results, we introduce the following hypotheses .

(A1) There exist two constants α and $\gamma_j > 0, j = 0,1,\cdots,k$ such that, for any $u_1, u_2 \in C(J,\mathbb{R})$

$$| G(u_1)) - G(u_2)) | \leq \alpha |u_1 - u_2|$$

and

$$\left|D^j(u_1) - D^j(u_2)\right| \leq \gamma_j |u_1 - u_2|$$

we assume that the non-linear terms $G(u(x))$ and $D^j(u) = \left(\frac{d^j}{dx^j}\right) u(x) = \sum_{i=0}^{\infty} \gamma_{i_j}$, (D^j is a derivative operator), $j = 0,1,\cdots,k$, are Lipschitz continuous .

(A2) We suppose that for all $a \leq t \leq x \leq b$, and $j = 0,1,\cdots,k$

$$\left|\frac{\lambda(x-t)^k K(x,t)}{k!\, p_k(x)}\right| \leq \theta_1, \quad \left|\frac{\lambda(x-t)^k K(x,t)}{k!}\right| \leq \theta_2,$$
$$\left|\frac{(x-t)^{k-1} p_j(t)}{(k-1)!\, p_k(t)}\right| \leq \theta_3, \quad \left|\frac{(x-t)^{k-1} p_j(t)}{(k-1)!}\right| \leq \theta_4,$$

(A3) There exist three functions θ_3^*, θ_4^*, and $\gamma^* \in C(D,\mathbb{R}^+)$, the set of all positive function continuous on $D = \{(x,t) \in \mathbb{R} \times \mathbb{R}: 0 \leq t \leq x \leq 1\}$ such that .

$$\theta_3^* = \max|\theta_3|, \ \theta_4^* = \max|\theta_4|, \text{ and } \gamma^* = \max\left|\gamma_j\right|$$

(A4) $\Psi(x)$ is bounded function for all x in $J = [a,b]$.

Theorem 3.1: Assume that (A1)-(A4) hold. If

$$0 < \psi = (\alpha\theta_1 + k\gamma^*\theta_3^*)(b-a) < 1 \qquad (21)$$

Then there exists a unique solution $u(x) \in C(J)$ to IVP $(1) - (2)$.

Proof: Let u_1 and u_2 be two various solutions of IVP $(1) - (2)$ then

$$|u_1 - u_2| = \left| \int_a^b \frac{\lambda(x-t)^k K(x,t)}{p_k(x)k!}\left[G(u_1) - G(u_2)\right)\right] dt$$
$$-\sum_{j=0}^{k-1} \int_a^b \frac{(x-t)^{k-1} p_j(t)}{p_k(t)(k-1)!}\left[D^j(u_1) - D^j(u_2)\right)\Bigg] dt \Big|$$
$$\leq \int_a^b \left|\frac{\lambda(x-t)^k K(x,t)}{p_k(x)k!}\right| \left\| G(u_1) - G(u_2) \right) \Big| \, dt$$
$$-\sum_{j=0}^{k-1} \int_a^b \left|\frac{(x-t)^{k-1} p_j(t)}{p_k(t)(k-1)!}\right| \left\| D^j(u_1) - D^j(u_2) \right) \Big| \, dt$$
$$\leq (\alpha\theta_1 + k\gamma^*\theta_3^*)(b-a)|u_1 - u_2|,$$

we obtain $(1-\psi)|u_1 - u_2| \leq 0$. Since $0 < \psi < 1$, so $|u_1 - u_2| = 0$, consequently, $u_1 = u_2$ and the proof is done.

Illustrative Example

Example 4.1: Consider the following Fredholm integro-differential equation .

$$u'(x) = 1 - \frac{1}{3}x + \int_0^1 xtu(t)dt \tag{22}$$

with the initial condition

$$u(0) = 0 \tag{23}$$

and the exact solution is $u(x) = x$.

Table 1. Comparison Results of the Example 3.1

x	Exact solution	$ADM_{n=16}$	$VIM_{n=10}$	$MADM_{n=12}$	$HPM_{n=9}$
0.1	0.1000000000	0.0999999205	0.0999999205	0.0999999205	0.0999999205
0.2	0.2000000000	0.1999996821	0.1999996821	0.1999996821	0.1999996821
0.3	0.3000000000	0.2999992847	0.2999992847	0.2999992847	0.2999992847
0.4	0.4000000000	0.3999987284	0.3999987284	0.3999987284	0.3999987284
0.5	0.5000000000	0.4999980132	0.4999980132	0.4999980132	0.4999980132
0.6	0.6000000000	0.5999971390	0.5999971390	0.5999971390	0.5999971390
0.7	0.7000000000	0.6999961058	0.6999961058	0.6999961058	0.6999961058
0.8	0.8000000000	0.7999949137	0.7999949137	0.7999949137	0.7999949137
0.9	0.9000000000	0.8999935627	0.8999935627	0.8999935627	0.8999935627
1.0	1.0000000000	0.9999920527	0.9999920527	0.9999920527	0.9999920527

The approximate solutions for the example 4.1 outcomes from the four above approaches are compared in the above table".

4. Conclusion

We covered four distinct approaches to solving integral and integro-differential equations, using a test case that has an exact solution known. The study illustrates the salient features of various approaches and offers some insight into which approach is preferred over the others. The aforementioned techniques have been effectively applied in this article to derive the approximateXsolutions of integral and integro-differential equations. The four approaches are effective, potent, and provide more accurate approximations. If closed-form solutions are available, they can also generate them. Despite the fact that both techniques get about the same results when appliedZto Volterra integro-differentialZequations. The simplest approach is HPM, which is also more suitable than the others".

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