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Generalized Variational Iteration Method for Solve Fractional Differential Equations

Zainab Sami Mahmood, Muayyad Mahmood Khalil*
College of Education for Pure Sciences, Tikrit University, Tikrit, Iraq
*Correspondence: medomath80@gmail.com

Abstract: In recent years, numerical techniques for solving fractional differential equations (FDEs) have gained significant attention due to their applications in various fields. One method that has shown promise is the Variational Iteration Method (VIM). However, challenges remain in addressing boundary value problems (BVPs) with high accuracy. To address this gap, this study introduces the Generalized Variational Iteration Method (GVIM), an extension of VIM specifically designed for BVPs. The proposed method delivers highly accurate solutions with evenly distributed errors across the domain. Numerical results demonstrate the method's superior convergence and reliability compared to existing techniques, offering a robust tool for solving complex FDEs in scientific and engineering applications.

Keywords: Fractional, Variational, Iteration, Absolute, Boundary, Generalized

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1. Introduction

The Variational Iteration Method (VIM), which has been created and examined by numerous researchers, is one of the numerous numerical techniques that have been used in recent years to achieve approximate answers [1–18]. Numerous numerical collocation techniques result in extremely discrete approximate solutions, which are nonetheless desirable but less desirable than analytical or semi-analytical answers. As an illustration, consider the wavelet-based approaches [19], spline collocation methods [20], Taylor collocation method [21], and Chebyshev collocation method [22]. Fractional differential equations have been the focus of many researchers in the past two decades.

It is believed that the application of fractional calculus goes back to 1823 when Abel provided the first application for fractional calculus with his tautochrone problem. The problem deals with the determination of an object such that the time it takes to fall under the influence of gravity is independent of the starting position [23]. Applications of fractional calculus are found in nearly all fields of study, including engineering, finance and economics, health sciences, and natural and social sciences [24–27]. Approximate solutions for fractional differential equations can be obtained by modifying the majority of numerical techniques used for ordinary differential equations. For example, fractional Initial Value Problems (IVPs) were solved by Abdulaziz et al. [28] using the Homotopy Perturbation Method (HPM).

In order to solve different linear and nonlinear fractional differential equations, the variational iteration method was changed from its original form [29, 30]. In order to solve fractional Bratu-type equations, Sakar et al. [31] proposed a Legendre reproducing kernel technique (L-RKM) that produced extremely accurate approximations. The Quasi-Newton's Method (QNM) [32], the Bezier Curve Method (BCM) [33], and the Sinc-Galerkin Method [34] are other techniques that have been employed to solve fractional differential equations. See [35] and its references for other techniques on solving fractional differential equations. This study extends the variational iteration method to BVPs, presenting a recently developed iterative approach for solving numerous fractional differential equations [36]. By contrasting it with alternative approaches, the process' accuracy and efficiency will be evaluated".

2. Materials and Methods

It is significant to remember that the right endpoint, x=b, is not included in the VIM, but the left endpoint, x=a, is. The VIM is hence effective for initial value issues (IVPs). When working with BVPs, the VIM is regarded as a drawback because it compromises the solution's correctness. We will add both ends, x=b and x=a, to the corrective function for BVPs in light of this limitation. The Generalized Variational Method is the name given to this modification (GVIM). We examine the adapted correction functional for BVPs as follows .

$$u_{n+1}(t) = u_n(t) + \int_a^t \lambda_1(s;t)[L(u_n)_s + N\tilde{u}_n(s) - f(s)]ds$$
$$+ \int_t^b \lambda_2(s;t)[L(u_n)_s + N\tilde{u}_n(s) - f(s)]ds, \tag{1}$$

where $\lambda_1(s;t)$ and $\lambda_2(s;t)$ are two general Lagrange multipliers defined on the intervals $[a,t]$ and $[t,b]$ respectively and satisfy the homogeneous BCs at $t = b$ and $t = a$ respectively. The $\tilde{u}_n$ is a restricted variation , i.e., $\delta\tilde{u}_n = 0$

Then, we will derive the accurate correctional functionals for the three kinds of BVPs .

(I) $u'' + f(t,u,u',u'') = 0$ where $a \le t \le b$ and BCs:

$$u(a) = \alpha, u(b) = \beta. \tag{2}$$

The correction function in (1) becomes

$$u_{n+1}(t) = u_n(t) + \int_a^t \lambda_1(s;t)\big[u_n''(s) + \tilde{f}(s, u_n(s), u_n'(s), u_n''(s))\big]ds \tag{3}$$

see that substituting $t = b$ on both sides of the equation makes the 2nd integral 0. Therefore, we need $\lambda_1(s;b) = 0$ to render the first integral to zero. A similar argument holds for λ_2. To find the best values of λ_1 and $\lambda_{2,}$ we integrate by parts two times the first term within each of the integrals in (3), and we obtain ,

$$\begin{aligned}\int_a^t \lambda_1(s;t)u_n''(s)ds &= \lambda_1(s;t)u_n'(s)|_a^t - \lambda_1'(s;t)u_n(s)|_a^t \\ &+ \int_a^t \lambda_1''(s;t)u_n(s)ds + \lambda_1(t;t)u_n'(t) - \lambda_1'(t;t)u_n(t) \\ &-\lambda_1(a;t)u_n'(a) + \lambda_1'(a;t)u_n(a) + \int_a^t \lambda_1''(s;t)u_n(s)ds\end{aligned}$$

In some way

$$\begin{aligned}\int_t^b \lambda_2(s;t)u_n''(s)ds &= \lambda_2(s;t)u_n'(s)|_t^b - \lambda_2'(s;t)u_n(s)|_t^b + \int_t^b \lambda_2''(s;t)u_n(s)d \\ &= \lambda_2(b;t)u_n'(b) - \lambda_2'(b;t)u_n(b) - \lambda_2(t;t)u_n'(t) \\ &+\lambda_2'(t;t)u_n(t) + \int_t^b \lambda_2''(s;t)u_n(s)ds.\end{aligned}$$

Substitute them in (3) we have

$$\begin{aligned}u_{n+1}(t) &= [1 - \lambda_1'(t;t) + \lambda_2'(t;t)]u_n(t) + [\lambda_1(t;t) - \lambda_2(t;t)]u_n'(t) \\ &-\lambda_1(a;t)u_n'(a) + \lambda_1'(a;t)u_n(a) + \lambda_2(b;t)u_n'(b) - \lambda_2'(b;t)u_n(b) \\ &+ \int_a^t \lambda_1''(s;t)u_n(s)ds + \int_t^b \lambda_2''(s;t)u_n(s)ds \\ &+ \int_a^t \lambda_1(s;t)\tilde{f}(s, u_n(s), u_n'(s), u_n''(s))ds \\ &+ \int_t^b \lambda_2(s;t)\tilde{f}(s, u_n(s), u_n'(s), u_n''(s))ds.\end{aligned}$$

Then, we take the variation concerning $u_n(t)$ on both sides of the above equation. Considering that $\tilde{f}$ is a restricted variation i.e $\delta\tilde{f}(s, u_n(s), u_n'(s), u_n''(s)) = 0$, we obtain .

$$\delta u_{n+1}(t) = [1 - \lambda_1'(t;t) + \lambda_2'(t;t)]\delta u_n(t) + [\lambda_1(t;t) - \lambda_2(t;t)]\delta u_n'(t)$$

"set the variation $\delta u_{n+1}(t) = 0$ to obtain the following stationary conditions".

$$\begin{aligned}
&1 - \lambda_1'(s;t) + \lambda_2'(s;t)|_{s=t} = 0,\\
&\lambda_1(s;t) - \lambda_2(s;t)|_{s=t} = 0,\\
&\lambda_1''(s;t) = 0,\ a \le s \le t,\\
&\lambda_2''(s;t) = 0,\ t \le s \le b.
\end{aligned}$$

Solving the above system of equations with the fact that $\lambda_1(s;a) = 0$ and $\lambda_2(s;b) = 0$ implies,

$$\lambda_1(s;t) = \frac{t-b}{a-b}s + \frac{a(b-t)}{a-b},\ a \le s \le t,$$

and

$$\lambda_2(s;t) = \frac{t-a}{a-b}s + \frac{b(a-t)}{a-b},\ t \le s \le b.$$

Thus, the GVIM for this case is

$$u_{n+1}(t) = u_n(t) + \int_a^t \left(\frac{t-b}{a-b}s + \frac{a(b-t)}{a-b}\right)[u_n''(s) + \tilde{f}(s, u_n(s), u_n'(s), u_n''(s))]ds$$

(II) $u'' - k^2u + f(t, u, u', u'') = 0$

where $a \le t \le b$ and subject to the BCs:

$$u(a) = \alpha, u(b) = \beta. \tag{4}$$

The correction function in (1) becomes

$$u_{n+1}(t) = u_n(t) + \int_a^t \lambda_1(s;t)[u_n''(s) - k^2u_n(s) + \tilde{f}(s, u_n(s), u_n'(s), u_n''(s))]ds \ (5)$$

Similar to case (I), substituting $t = b$ on both sides of the equation makes the second integral zero. Thus, we require $\lambda_1(s; b) = 0$ to render the first integral to zero . A similar argument holds for λ_2.

To find the optimal values of λ_1 and $\lambda_{2,}$ we integrate by parts twice the first term within each of the integrals in (5) obtaining ,

$$\int_a^t \lambda_1(s;t)u_n''(s)ds = \lambda_1(s;t)u_n'(s)|_a^t - \lambda_1'(s;t)u_n(s)|_a^t + \int_a^t \lambda_1''(s;t)u_n(s)ds$$
$$= \lambda_1(t;t)u_n'(t) - \lambda_1'(t;t)u_n(t) - \lambda_1(a;t)u_n'(a)$$
$$+\lambda_1'(a;t)u_n(a) + \int_a^t \lambda_1''(s;t)u_n(s)ds.$$

Similarly,

$$\int_t^b \lambda_2(s;t)u_n''(s)ds = \lambda_2(s;t)u_n'(s)|_t^b - \lambda_2'(s;t)u_n(s)|_t^b + \int_t^b \lambda_2''(s;t)u_n(s)ds$$
$$= \lambda_2(b;t)u_n'(b) - \lambda_2'(b;t)u_n(b) - \lambda_2(t;t)u_n'(t)$$
$$+\lambda_2'(t;t)u_n(t) + \int_t^b \lambda_2''(s;t)u_n(s)ds.$$

Replacing them in (5) yields,

$$u_{n+1}(t) = [1 - \lambda_1'(t;t) + \lambda_2'(t;t)]u_n(t) + [\lambda_1(t;t) - \lambda_2(t;t)]u_n'(t)$$
$$-\lambda_1(a;t)u_n'(a) + \lambda_1'(a;t)u_n(a) + \lambda_2(b;t)u_n'(b) - \lambda_2'(b;t)u_n(b)$$
$$+ \int_a^t [\lambda_1''(s;t) - k^2\lambda_1(s;t)]u_n(s)ds + \int_t^b [\lambda_2''(s;t) - k^2\lambda_2(s;t)]u_n(s)ds$$
$$+ \int_a^t \lambda_1(s;t)\tilde{f}(s, u_n(s), u_n'(s), u_n''(s))ds$$
$$+ \int_t^b \lambda_2(s;t)\tilde{f}(s, u_n(s), u_n'(s), u_n''(s))ds.$$

Next, we take the variation concerning $u_n(t)$ on both sides of equation (5).

Taking into account that $\tilde{f}$ is a restricted variation, i.e $\delta\tilde{f}(s, u_n(s), u_n'(s), u_n''(s)) = 0$, we obtain.

$$\delta u_{n+1}(t) = [1-\lambda_1'(t;t)+\lambda_2'(t;t)]\delta u_n(t) + [\lambda_1(t;t)-\lambda_2(t;t)]\delta u_n'(t)$$
$$+\delta\left(\int_a^t [\lambda_1''(s;t)-k^2\lambda_1(s;t)]u_n(s)ds\right)$$

We set the variation $\delta u_{n+1}(t) = 0$ to obtain the following stationary conditions

$$\begin{aligned} &1-\lambda_1'(s;t)+\lambda_2'(s;t)|_{s=t}=0,\\ &\lambda_1(s;t)-\lambda_2(s;t)|_{s=t}=0,\\ &\lambda_1''(s;t)-k^2\lambda_1(s;t)=0,\ a\le s\le t,\\ &\lambda_2''(s;t)-k^2\lambda_2(s;t)=0,\ t\le s\le b.\end{aligned}$$

Solving the above system of equations with the fact that $\lambda_1(s;a) = 0$ and $\lambda_2(s;b) = 0$ gives us

$$\lambda_1(s;t) = \frac{\sinh(k(b-t))\sinh(k(a-s))}{k\sinh(k(a-b))},\ a\le s\le t$$

and

$$\lambda_2(s;t) = \frac{\sinh(k(a-t))\sinh(k(b-s))}{k\sinh(k(a-b))},\ t\le s\le b$$

Thus, the GVIM for this case is:

$$u_{n+1}(t) = u_n(t) + \int_a^t \left(\frac{\sinh(k(b-t))\sinh(k(a-s))}{k\sinh(k(a-b))}\right) + \int_t^b \left(\frac{\sinh(k(a-t))\sinh(k(b-s))}{k\sinh(k(a-b))}\right). \left[u_n''(s)-k^2u_n+\tilde{f}(s,u_n(s),u_n'(s),u_n''(s))\right]ds$$

(III) $u''' + f(t,u,u',u'',u''') = 0$

Where $a\le t\le b$ and subject to the BCs:

$$u(a)=\alpha, u'(a)=\beta, u(b)=\eta. \tag{6}$$

The correction function in (1) becomes

$$u_{n+1}(t) = u_n(t) + \int_a^t \lambda_1(s;t)\left[u_n'''(s)+\tilde{f}\left(s,u_n(s),u_n'(s),u_n''(s),u'''(s)\right)\right]ds \tag{7}$$

The second integral becomes zero by substituting t=b on both sides of the equation, just like in the preceding situations. Therefore, in order to make the first integral zero, we need λ_1 (s;b)=0. For λ_2, a similar reasoning applies. In order to determine the ideal values for λ_1 and λ_2 we split the first term in each of the integrals in (7) into three parts, resulting in ,

$$\begin{aligned}
\int_a^t \lambda_1(s;t)u_n'''(s)ds &= \lambda_1(s;t)u_n''(s)|_a^t - \lambda_1'(s;t)u_n'(s)|_a^t + \lambda_1''(s;t)u_n|_a^t \\
&\quad - \int_a^t \lambda_1'''(s;t)u_n(s)ds \\
&= \lambda_1(t;t)u_n''(t) - \lambda_1'(t;t)u_n'(t) + \lambda_1''(t;t)u_n(t) \\
&\quad -\lambda_1(a;t)u_n''(a) + \lambda_1'(a;t)u_n'(a) - \lambda_1''(a;t)u_n(a) \\
&\quad - \int_a^t \lambda_1'''(s;t)u_n(s)ds
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_t^b \lambda_2(s;t)u_n'''(s)ds &= \lambda_2(s;t)u_n''(s)|_t^b - \lambda_2'(s;t)u_n'(s)|_t^b + \lambda_2''(s;t)u_n|_t^b \\
&\quad - \int_t^b \lambda_2'''(s;t)u_n(s)ds \\
&= \lambda_2(b;t)u_n''(b) - \lambda_2'(b;t)u_n'(b) + \lambda_2''(b;t)u_n(b) \\
&\quad -\lambda_2(t;t)u_n''(t) + \lambda_2'(t;t)u_n'(t) - \lambda_2''(t;t)u_n(t) \\
&\quad - \int_t^b \lambda_2'''(s;t)u_n(s)ds
\end{aligned}$$

Replacing them back in (7) yields,

$$u_{n+1}(t) = [1+\lambda_1''(t;t) - \lambda_2''(t;t)]u_n(t) + [-\lambda_1'(t;t) + \lambda_2'(t;t)]u_n'(t)$$
$$+[\lambda_1(t;t) - \lambda_2(t;t)]u_n''(t) - \lambda_1(a;t)u_n''(a) + \lambda_1'(a;t)u_n'(a)$$
$$-\lambda_1''(a;t)u_n(a) + \lambda_2(b;t)u_n''(b) - \lambda_2'(b;t)u_n'(b) + \lambda_2''(b;t)u_n(b)$$
$$-\int_a^t \lambda_1'''(s;t)u_n(s)ds - \int_2^{'''} (s;t)u_n(s)ds$$
$$+\int_a^t \lambda_1(s;t)\tilde{f}(s,u_n(s),u_n'(s),u_n''(s),u_n'''(s))ds$$
$$+\int_t^b \lambda_2(s;t)\tilde{f}(s,u_n(s),u_n'(s),u_n''(s),u_n'''(s))ds.$$

Next, we take the variation with respect to $u_n(t)$ of both sides of the above equation. Taking into account that $\tilde{f}$ is a restricted variation i.e

$\delta f(s,u_n(s),u_n'(s),u_n''(s),u'''(s)) = 0$, we obtain

$$u_{n+1}(t) = [1+\lambda_1''(t;t) - \lambda_2''(t;t)]\delta u_n(t) + [-\lambda_1'(t;t) + \lambda_2'(t;t)]\delta u_n'(t)$$
$$+[\lambda_1(t;t) - \lambda_2(t;t)]\delta u_n''(t) - \delta\left(\int_a^t \lambda_1'''(s;t)u_n(s)ds\right)$$

We set the variation $\delta u_{n+1}(t) = 0$ to obtain the following stationary conditions .

$$1+\lambda_1''(s;t) - \lambda_2''(s;t)|_{s=t} = 0,$$
$$-\lambda_1'(s;t) + \lambda_2'(s;t)|_{s=t} = 0,$$
$$\lambda_1(s;t) - \lambda_2(s;t)|_{s=t} = 0,$$
$$\lambda_1'''(s;t) = 0,\ a \le s \le t,$$
$$\lambda_2'''(s;t) = 0,\ t \le s \le b.$$

Solving the above system of equations with the fact that $\lambda_1(s;a) = 0$ and $\lambda_2(s;b) = 0$ gives us .

$$\lambda_1(s;t) = \frac{1}{2}\frac{(a-s)(b-t)}{(a-b)^2}(ab - 2as + at + bs - 2bt + st),\ a \le s \le t$$

and

$$\lambda_2(s;t) = \frac{1}{2}\frac{(a-t)^2(b-s)^2}{(a-b)^2}, t \le s \le b.$$

Thus, the GVIM for this case is:

$$u_{n+1}(t) = u_n(t) + \int_a^t \left(\frac{1}{2}\frac{(a-s)(b-t)}{(a-b)^2}(ab - 2as + at + bs - 2bt + st)\right)$$
$$\left[u_n'''(s) + \tilde{f}(s, u_n(s), u_n'(s), u_n''(s), u_n'''(s))\right]ds$$

3. Results and Discussion

Example 3.1:

Consider the following fractional nonlinear differential equation, which is taken from [33]

$$u^{1.5}(x) + u^3 - \frac{\Gamma(2.9)}{\Gamma(1.4)}x^{0.4} + (x^{1.9} - 1)^3 = 0, \tag{8}$$

where $0 \le x \le 1$ and subject to the following boundary conditions (BCs)

$$u(0) = -1,\ u(1) = 0. \tag{9}$$

The exact solution is $u(x) = x^{1.9} - 1$. The iteration formula for the GVIM with $a = 0$, $b = 1$ yields .

$$u_{n+1} = u_n + \int_0^x (1-x)s\left[u^\alpha + u^3 - \frac{\Gamma(2.9)}{\Gamma(1.4)}s^{0.4} + (s^{1.9} - 1)^3\right]ds$$
$$+ \int_x^1 (1-s)x\left[u^\alpha + u^3 - \frac{\Gamma(2.9)}{\Gamma(1.4)}s^{0.4} + (s^{1.9} - 1)^3\right]ds$$

where $\alpha = 1.5$ and $u_0 = x^{1.5} - 1$ is the initial iterate .

Table 1. Absolute error for Example 3.1 using GVIM

t	Absolute error
0.1	0.03037
0.2	0.04303
0.3	0.04515
0.4	0.04032
0.5	0.03157
0.6	0.02156
0.7	0.01237
0.8	0.00541
0.9	0.00131

Table 3.1's initial iteration results guarantee the GVIM strategy's simplicity, efficacy, and increased accuracy. We can conclude that the convergence average is rapid since the findings shown above, which were acquired after a single iteration, are accurate

Example 3.2:

Consider the following fractional linear differential equation, which is taken from [32]

$$u^{1.2}(x) + \frac{3}{57}u(x) - x - \frac{3x^{2.2}}{57\Gamma(3.2)} = 0, \tag{10}$$

subject to the following nonhomogeneous boundary conditions (BCs)

$$u(0) = 0,\ u(1) = \frac{1}{\Gamma(3.2)} \tag{11}$$

The exact solution is $u(x) = \frac{x^{2.2}}{\Gamma(3.2)}$. The iteration formula for the GVIM with $a = 0$, $b = 1$, and $\alpha = 1.2$ yields ,

$$u_{n+1} = u_n + \int_0^x (1-x)s\left[u^\alpha + \frac{3}{57}u(x) - s - \frac{3s^{2.2}}{57\Gamma(3.2)}\right]ds$$
$$+ \int_x^1 (1-s)x\left[u^\alpha + \frac{3}{57}u(x) - s - \frac{3s^{2.2}}{57\Gamma(3.2)}\right]ds$$

Table 2. Absolute error for Example 3.2 using GVIM

t	Absolute error
0.1	$6.67230(-5)$
0.2	$7.48105(-5)$
0.3	$9.96561(-5)$
0.4	$5.53723(-7)$
0.5	$2.24204(-4)$
0.6	$2.87651(-4)$
0.7	$1.50894(-4)$
0.8	$3.17525(-5)$
0.9	$9.92607(-5)$

The 50^{th} iterate results in Table 3.2 ensure that the GVIM approach is efficient .

Example 3.3:

Consider the following fractional differential equation .

$$u^{\alpha}(x)+u(x)-x^5+x^4-x^{3.5}\frac{128}{7\sqrt{\pi}}+x^{2.5}\frac{64}{5\sqrt{\pi}}=0, \tag{12}$$

subject to the following nonhomogeneous boundary conditions (BCs)

$$u(0)=0,\ u(1)=0 \tag{13}$$

The exact solution is $u(x)=x^5-x^4$. The iteration formula for the GVIM with $a=0$, $b=1$ yields ,

$$u_{n+1}=u_n-\int_0^x(1-x)s\left[u^{\alpha}+u(x)-s^5+s^4-s^{3.5}\frac{128}{7\sqrt{\pi}}+s^{2.5}\frac{64}{5\sqrt{\pi}}\right]ds$$
$$-\int_x^1(1-s)x\left[u^{\alpha}+u(x)-s^5+s^4-s^{3.5}\frac{128}{7\sqrt{\pi}}+s^{2.5}\frac{64}{5\sqrt{\pi}}\right]ds$$

where $\alpha=1.5$ and $u_0=x^4(x-1)$ is the initial iterate which is equal to the exact solution .

Table 3. Absolute error for Example 3.3 using GVIM

t	Absolute error
0.1	$1.79100(-41)$

The 1st iterate results in Table 3.3 ensure that GVIM technique is simple and accurate .

4. Conclusion

This paper presents a novel method for the numerical solution of different Fractional Boundary Value Problems (FBVPs) that is based on expanding the Variational Iteration Method (VIM). For the purpose of solving BVPs, the suggested Generalized Variational Iteration Method (GVIM) is especially appropriate. The findings obtained from eight distinct cases confirm that the suggested method has a high rate of convergence and great accuracy when compared to other methods. It produces errors that are nearly evenly distributed over the interval.

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